# Lecture notes on Integrable Systems and Friezes 

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#### Abstract

The notion of integrability, which goes back to Euler and Jacobi, is a central one in mathematics. Integrable systems appear in many areas of mathematics, unifying diverse ideas in algebra, geometry, and physics. The course will focus on particular systems exhibiting mutation dynamics as in the recent theory of cluster algebras of Fomin and Zelevinsky. Frieze patterns defined by Coxeter in the 1970s give a simple concrete exemple of such systems. The course will present recent developments around the notion of friezes in connection with representation theory and cluster integrability.


## Foreword

These are notes of a series of lectures on Integrable Systems and Friezes given at the LMS-CMI Research School "New trends in representation theory - The impact of cluster theory in representation theory", held at the University of Leicester 19-23 June 2017. I would like to thank again the organizers Karin Baur and Sibylle Schroll for their kind invitation.

I am very grateful to Joe Pallister who provided me with a typed version of the notes for the first three lectures, and to Max Glick for the exercise sheet and tutorial assistance during the school. I would like to thank also Valentin Ovsienko for his numerous valuable advice during the preparation of the lectures and for his careful reading of the final version of the notes.

For the students of the research school I suggested a first list of references of introductory texts on the notions covered in the lectures, trying to select mainly books, survey articles or easy-to-read papers. In these notes I have included more references for recent developments on the subjects. Here was the preliminary list:

Lecture 1: Discrete dynamical/integrable systems
Typical example: billiards [1]
Liouville-Arnold integrability [2]
Example 0: Pentagon recurrence (Gauss map) [3]
Example 1: Pentagram map [4]
Example 2: Somos-4 recurrence [5]
Example 3: Octahedron recurrence ( $T$-systems) [6]
Lecture 2: Cluster Algebras, cluster dynamics, I [7]
Basics on cluster algebras [8], [9]
Cluster compatible Poisson brackets [10]
Back to Pentagram map [11]
Lecture 3: Cluster Algebras, cluster dynamics, II
Cluster structure and integrability of Somos-4 [12], [9]
T-systems and Zamolodchikov periodicity [13], [6]

Lecture 4: Coxeter Friezes and first generalizations, [14]
Definition and some nice properties [15], [16]
The variant of 2-friezes [17]
$S L_{r}$-friezes [18]
Lecture 5: Friezes and Representation Theory [14]
Friezes defined over a quiver [19]
Quiver representations [20]
Some open problems
In these notes several important results are stated as Propositions (that can be checked as exercises) or Theorems without references. They are assumed to be "classical results" and the reader can find proofs and exact references in the above mentioned texts that are recall at the beginning of each sections. My apologies for not citing there the original work of the authors of the results.

## 1 Lecture 1: Discrete dynamical/integrable systems

In this lecture we define the notion of Liouville-Arnold integrabilty and present four examples of dynamical systems which will be studied in the next lectures using ingredients of the theory of cluster algebras.

### 1.1 Introductory example: Billiards [1]

We start with an informal example to get some ideas and images of what is a discrete dynamical/integrable system.


Consider a billiard table. The ball moves along straight lines and when hitting the boundary it reflects according to the natural law: the angle of incidence equals the angle of reflection (the above picture could be not really accurate). The ball is a point, it has no mass and no speed: it will move forever (we just consider the geometric problem).

The question is to understand the geometric structure and the asymptotic behavior of the system. Would it stabilize between few points, would it go back to the intial points...? Drawing the picture on the billiard table will not be very useful. One prefers to draw trajectories in the "phase space".

We have two parameters for this system: $x \in \mathbb{P}^{1}$ is the position of the ball on the boundary, $y \in \mathbb{P}^{1}$ is the outgoing direction (no speed).

When the system is integrable the phase space is "well organised", otherwise it is rather "untidy".


If the billiard table is an ellipse, then the billiard is integrable; the converse statement is the classical Birkhoff conjecture. This example shows how rare the integrable systems are among more general (ergodic) dynamical systems. Let us also mention that integrability of the elliptic billiard is an efficient way to (re)prove some classical geometric theorems, such as the Poncelet theorem.

### 1.2 Liouville-Arnold Integrability [2]

The system is the map $\Phi: T \times M \rightarrow M$ where $T$ is time (for discrete time $T=\mathbb{N}$ ) and $M$ is the phase space, an algebraic variety with $\operatorname{dim}(M)=n$. The value $x_{k}=\Phi\left(k, x_{0}\right)$ gives the state of the system at time $k$ starting at $x_{0}$. The actual time has no effect on the stae, we can simplify this as

$$
\varphi: M \rightarrow M, \quad \varphi^{k}\left(x_{0}\right)=x_{k}
$$

Let $K(M)$ be the field of rational functions over $M$ equipped with a Poisson bracket

$$
\{-,-\}: K(M) \times K(M) \rightarrow K(M)
$$

that is assumed to be $\varphi$-invariant, i.e.

$$
\{f \circ \varphi, g \circ \varphi\}=\{f, g\} \circ \varphi, \quad \forall f, g \in K(M) .
$$

We denote the dimension of the kernel of $\{-,-\}$ as $s \geq 0$.
Definition. The system is said to be completely integrable in the sense of Liouville-Arnold if there exist

$$
C_{1}, \ldots, C_{s}, F_{1}, \ldots, F_{r} \in K(M)
$$

satisfying the conditions:

- the functions are algebraically independent,
- the functions are $\varphi$-invariant (they are called "conserved quantities" or "first integrals"),
- $\left\{C_{i}, f\right\}=0, \quad \forall f \in K(M) \quad$ (the $C_{i}$ are called "Casimirs"),
- $\left\{F_{i}, F_{j}\right\}=0, \quad \forall i, j \in\{1,2, \ldots, r\} \quad$ (the $F_{i}$ 's "Poisson commute"),
- $n=2 r+s \quad$ ("enough conserved quantities").

Remark. If $\{-,-\}$ is non-degenerate (i.e. $s=0$ ) then $n=2 r$ and the integrable system will have a tori foliation

$$
\mathbb{T}=\underbrace{\mathbb{P}^{1} \times \mathbb{P}^{1} \times \ldots \times \mathbb{P}^{1} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{r}
$$

such that

- the tori are $\varphi$-invariant,
- there is linear motion on $\mathbb{T}$ (picture here),
- $\left.F_{i}\right|_{\mathbb{T}}=$ constant.


### 1.3 Examples

There are many classical examples of integrable systems due to Newton, Euler, Jacobi, ... We introduce here very particular examples that we will study in the next lectures using the combinatorics of cluster algebras.

## Pentagonal recurrence or Gauss map [3].

The pentagonal recurrence or Gauss map is given by

$$
x_{n+2} x_{n}=x_{n+1}+1 .
$$

The sequence $\left(x_{n}\right)$ is determined by two consecutive values, e.g. ( $x_{1}, x_{2}$ ). This leads to a map $\left(x_{1}, x_{2}\right) \mapsto\left(x_{2}, x_{3}\right)=\left(x_{2},\left(1+x_{2}\right) / x_{1}\right)$, let us denote

$$
\varphi:\binom{x}{y} \mapsto\binom{y}{\frac{1+y}{x}}
$$

where $(x, y) \in M=\mathbb{R}^{2}$ or $\mathbb{C}^{2}$ and $K(M)=\mathbb{R}(x, y)$ or $\mathbb{C}(x, y)$. The dimension of $M$ is 2 . We can define a Poisson bracket as

$$
\{f, g\}=x y\left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y}-\frac{\partial f}{\partial y} \frac{\partial g}{\partial x}\right)
$$

but it is actually enough to just define $\{x, y\}=x y$.
Proposition 1.1. One has

1. $\{-,-\}$ is $\varphi$-invariant
2. $F_{1}$ and $F_{2}$ are $\varphi$-invariant, where

$$
F_{1}:=\frac{(1+x)(1+x+y)(1+y)}{x y}, \quad F_{2}:=x+y+\frac{2+x}{y}+\frac{2+y}{x}+\frac{1}{x y}
$$

3. $\varphi$ is 5 periodic.

This means that $\varphi$ is LA integrable. Actually one can find more invariant functions than needed (one only needs one), it is sometimes called "super integrable". The periodic behavior of the system is not a general behavior of integrable systems.

Exercise 1.2. Prove Proposition 1.1

## The pentagram map [4].

The pentagram map $T$ is a dynamical system introduced by Richard Schwartz in 1992. The pentagram map acts on the space of $n$-gons in the projective plane modulo projective equivalence. Given an $n$-gon $P$, the corresponding $n$-gon $T(P)$ is the convex hull of the intersection points of consecutive shortest diagonals of $P$, see Figure 1.


Figure 1: The pentagram map.
More precisely $T$ acts on the moduli space of closed $n$-gons is

$$
\begin{equation*}
\mathcal{C}_{n}:=\left\{\left(v_{i}\right)_{i \in \mathbb{Z}}: v_{i} \in \mathbb{P}^{2}, v_{i+n}=v_{i}\right\} / \mathbb{P} S L_{3} . \tag{1}
\end{equation*}
$$

where we consider generic points, i.e. three consecutive $v_{i}$ are not on the same line. The map $T$ also acts on the larger space of twisted $n$-gons:

$$
\begin{equation*}
\mathcal{P}_{n}:=\left\{\left(v_{i}\right)_{i \in \mathbb{Z}}: v_{i} \in \mathbb{P}^{2}, \exists M \in \mathbb{P} S L_{3}: v_{i+n}=M v_{i}\right\} / \mathbb{P} S L_{3} . \tag{2}
\end{equation*}
$$

Note that $\mathcal{C}_{n}$ is of codimension 8 in $\mathcal{P}_{n}$, defined by the condition $M=\mathrm{Id}$.
We have ( $a, b$ )-coordinates on $\mathcal{P}_{n}$ defined as follows

$$
\mathcal{P}_{n} \ni\left(v_{i}\right) \xrightarrow{\text { lift }}\left(V_{i}\right) \in \mathbb{R}^{3}
$$

such that $\operatorname{det}\left(V_{i}, V_{i+1}, V_{i+2}\right)=1$. Then

$$
V_{i}=a_{i} V_{i-1}-b_{i} V_{i-2}+V_{i-3}
$$

gives $2 n$ coordinates $\left(a_{i}, b_{i}\right)$ for $i=1, \ldots, n$ (since $M v_{i}=v_{i+n}$ one has $a_{i+n}=a_{i}, b_{i+n}=b_{i}$ ).
Another system of parameters on $\mathcal{P}_{n}$ using cross ratios will be studied in the exercise session (see Problem 2 of the exercise sheet).

The Liouville-Arnold integrability of $T$ acting on $\mathcal{P}_{n}$ and $\mathcal{C}_{n}$ was established in [21], [22] using the ( $a, b$ )-coordinates.

In Lecture 3 we will see that the pentagram map is connected to the theory of cluster algebras and use cluster framework to discuss the integrabilty of the map.

Exercise 1.3. For $n=5, \mathcal{P}_{n}$ has 10 parameters $\left(a_{i}, b_{i}\right)$ for $i=1, \ldots, 5$. The space $\mathcal{C}_{5}$ is given by 8 equations. Express $a_{i}, b_{i}$ in terms of the two parameters $x:=b_{1}$ and $y:=a_{1}$.

## The Somos-4 recurrence [5].

The Somos-4 recurrence is given by

$$
x_{n+4} x_{n}=x_{n+1} x_{n+3}+x_{n+2}^{2} .
$$

With initial conditions $x_{1}=x_{2}=x_{3}=x_{4}=1$ one gets the following sequence:

$$
1,1,1,1,2,3,7,23,59,314,1529,8209,83313 \ldots
$$

which surprisingly contains only integers and no rational numbers as expected. This recurrence is part of a larger family of recurrences that are of some interest for number theorists.

It leads to the following dynamical system on $M=\mathbb{R}^{4}$ or $\mathbb{C}^{4}$ :

$$
\varphi:\left(\begin{array}{c}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right) \mapsto\left(\begin{array}{c}
x_{2} \\
x_{3} \\
x_{4} \\
\frac{x_{2} x_{4}+x_{3}^{2}}{x_{1}}
\end{array}\right)
$$

We will see in Lecture 2 that this is not Liouville-Arnold integrable but there is a reduction $\tilde{\varphi}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ that is.

## The octahedron recurrence / T-systems / discrete Hirota... [6].

This recurrence appears in various contexts: e.g. they are functional identities satisfied by the transfer matrices in solvable lattice model. They are also relations satisfied by the $q$-characters of KirillovReshetikhin modules of quantum affine algebras...

$$
T_{i-1, j, k} T_{i+1, j, k}-T_{i, j-1, k} T_{i, j+1, k}=T_{i, j, k-1} T_{i, j, k+1}
$$

for $i, j, k \in \mathbb{Z}$.
The above equation implies the vertices of an octahedron in the 3D space.


We will consider the restricted $T$-systems with boundary conditions:

$$
\begin{array}{lll}
T_{i, j, 0}=T_{i, j, r+1}=1, & T_{i, j, k}=0, & k \notin\{1, \ldots, r\}, \\
T_{i, 0, k}=T_{i, w+1, k}=1, & T_{i, j, k}=0, & j \notin\{1, \ldots, w\} . \tag{3}
\end{array}
$$

We will see in Lecture 3 that this system is somehow "super-integrable" as it was the case for the pentagonal recurrence

For $r=1$, the recurrence simplify to $T_{i-1, j} T_{i+1, j}-T_{i, j-1} T_{i, j+1}=1$ and we get a so-called Coxeter frieze.

Friezes will be studied in more details in Lecture 4 and 5.
Exercise 1.4. Complete the following piece of Coxeter frieze of width 2 (i.e. restricted $T$-system with $r=1, w=2$ ).


### 1.4 Solutions

## Solution of Exercise 1.2.

Call $\left(x^{\prime}, y^{\prime}\right):=\varphi(x, y)$. For question 1 and 2 , one needs to check $\left\{x^{\prime}, y^{\prime}\right\}=x^{\prime} y^{\prime}$ and
$\frac{\left(1+x^{\prime}\right)\left(1+x^{\prime}+y^{\prime}\right)\left(1+y^{\prime}\right)}{x^{\prime} y^{\prime}}=\frac{(1+x)(1+x+y)(1+y)}{x y}, x^{\prime}+y^{\prime}+\frac{2+x^{\prime}}{y^{\prime}}+\frac{2+y^{\prime}}{x^{\prime}}+\frac{1}{x^{\prime} y^{\prime}}=x+y+\frac{2+x}{y}+\frac{2+y}{x}+\frac{1}{x y}$,
by direct computations. For question 3, one computes the iterates and gets

$$
\binom{x}{y} \rightarrow\binom{y}{\frac{1+y}{x}} \rightarrow\binom{\frac{1+y}{x}}{\frac{1+x+y}{x y}} \rightarrow\binom{\frac{1+x+y}{x y}}{\frac{1+x}{y}} \rightarrow\binom{\frac{1+x}{y}}{x} \rightarrow\binom{x}{y}
$$

## Solution of exercise 1.3.

Let $v=\left(v_{i}\right)$ be a closed polygon and $V=\left(V_{i}\right)$ be the lift in the 3D space. Since we look at the polygons up to projective transformation, we can assume that

$$
V_{-2}=\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right), \quad V_{-1}=\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right), \quad V_{0}=\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right)
$$

The $(a, b)$-coordinates of $v$, viewed as a twisted polygon, are defined by the recurrence relation $V_{i}=$ $a_{i} V_{i-1}-b_{i} V_{i-2}+V_{i-3}$. So we get

$$
V_{1}=\left(\begin{array}{c}
1 \\
-b_{1} \\
a_{1}
\end{array}\right), \quad V_{2}=\left(\begin{array}{c}
a_{2} \\
-a_{2} b_{1}+1 \\
a_{1} a_{2}-b_{2}
\end{array}\right), \quad V_{3}=\left(\begin{array}{c}
a_{2} a_{3}-b_{3} \\
-a_{2} a_{3} b_{1}+a_{3}+b_{1} b_{3} \\
a_{1} a_{2} a_{3}-b_{2} a_{3}-a_{1} b_{3}+1
\end{array}\right), \ldots
$$

At this stage we can identify $V_{3}=V_{-2}$. This will lead to the first three equations, and will simplify the expressions for $V_{4}$ and $V_{5}$ using $V_{-2}$ instead of $V_{3}$ in the recurrence relation. We get

$$
V_{3}=\left(\begin{array}{c}
a_{2} a_{3}-b_{3} \\
-a_{2} a_{3} b_{1}+a_{3}+b_{1} b_{3} \\
a_{1} a_{2} a_{3}-b_{2} a_{3}-a_{1} b_{3}+1
\end{array}\right)=\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right), \quad V_{4}=\left(\begin{array}{c}
a_{4}-b_{4} a_{2}+1 \\
a_{2} b_{1} b_{4}-b_{1} \\
-a_{1} a_{2} b_{4}+b_{2} b_{4}+a_{1}
\end{array}\right)=\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right), \quad V_{5}=\left(\begin{array}{c}
-b_{5}+a_{2} \\
a_{5}-a_{2} b_{1}+1 \\
a_{1} a_{2}-b_{2}
\end{array}\right)=\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right)
$$

The equations can be solved in $x=b_{1}, y=a_{1}$. We obtain

$$
\begin{aligned}
& b_{1}=x, \quad a_{1}=y, \quad b_{2}=\frac{1+y}{x}, \quad a_{2}=\frac{1+x+y}{x y}, \quad b_{3}=\frac{1+x}{y}, \\
& =a_{3}, \quad=b_{4}, \quad=a_{4}, \quad=b_{5}, \quad=a_{5} .
\end{aligned}
$$

## Solution of exercise 1.4.

On obtains the Coxeter frieze of width 2 .

$$
\left.\begin{array}{ccccccccccccc} 
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\ldots & & y & & \frac{1+x+y}{x y} & & x & & \frac{1+y}{x} & & \frac{1+x}{y} & & y \\
& x & & \frac{1+y}{x} & & \frac{1+x}{y} & & y & & \frac{1+x+y}{x y} & & x & \\
& \ldots & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}\right)
$$

## Conclusion of the exercises.

The pentagonal recurrence, the space of closed pentagons and the the Coxeter frieze of width 2 are the same object. We will see that they are related to the cluster algebra of type $\mathbf{A}_{2}$.

## 2 Lecture 2: Cluster Algebras and Cluster Dynamics (I) [7]

### 2.1 Basics (not in full generality) [8]

Cluster algebras are commutative associative algebras defined by generators and relations. These are not given from the beginning but produced recursively.

The initial data is $\underline{x}=\left(x_{1}, \ldots, x_{n}\right)$, which are free variables, and $\Gamma$, which is a quiver without 1 or 2 loops and vertices labelled 1 to $n$. The pair $(\underline{x}, \Gamma)$ is called the initial seed. We produce more seeds by mutation. Mutation in direction $k$ produces the seed ( $\underline{x}^{\prime}, \Gamma^{\prime}$ ) defined as

$$
\mu_{k}\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}^{\prime}, \ldots, x_{n}^{\prime}\right)
$$

where $x_{i}^{\prime}=x_{i}$ if $i \neq k$ and

$$
x_{k}^{\prime}=\frac{1}{x_{k}}\left(\prod_{i \rightarrow k} x_{i}+\prod_{i \leftarrow k} x_{i}\right)
$$

The quiver mutation $\Gamma^{\prime}:=\mu_{k} \Gamma$ is obtained in the following way:

1. $\forall i \rightarrow k \rightarrow j$ add an arrow $i \rightarrow j$
2. Reverse all arrows touching $k$
3. Delete any two loops that may have appeared from step 1.

Remark. Seed mutation is an involution.
Example 2.1. Example of mutation
$\left(x_{1}, x_{2}, x_{3}\right)$

exchange graph here?)
In a seed $(\underline{x}, \Gamma) \underline{x}$ is called a cluster and its elements are called cluster variables.
Definition. The cluster algebra $\mathcal{A}(\Gamma)$ is the subalgebra of $\mathbb{Q}\left(x_{1}, \ldots, x_{n}\right)$ generated by all cluster variables created by all possible mutations from all seeds.

We now give some fundamental results.
Theorem 2.2 ( "Laurent Phenomenon"). All cluster variables belong to $\mathbb{Z}\left[x_{1}^{ \pm 1}, \ldots, x_{n}^{ \pm 1}\right]$.
Theorem 2.3 ("Positivity"). All cluster variables belong to $\mathbb{Z}_{\geq 0}\left[x_{1}^{ \pm 1}, \ldots, x_{n}^{ \pm 1}\right]$.
Theorem 2.4 ("Finite type classification"). There are a finite number of cluster variables if and only if $\Gamma$ is mutation equivalent to an orientation of a Dynkin diagram of type ADE.

Exercise 2.5. 1. Find the exchange graph for $\left(\left(x_{1}, x_{2}\right), 1 \rightarrow 2\right)$.
2. Consider a regular convex pentagon with sides of length 1 . Call $x_{1}, x_{2}$ the length of two of its diagonals. Express the lengths of the three remaining diagonals in terms of $x_{1}, x_{2}$ (hint: use the Ptolemy rule).


Ptolemy Rule: $\quad E F=A D+B C$


Remark. There are many possible generalizations for the definition of cluster algebras (e.g. frozen variables, skew-symmetrizable matrix instead of $\Gamma \ldots$...)

### 2.2 Y-patterns [8]

We will often identify the quiver $\Gamma$ and its adjancy matrix $B$ defined by

$$
b_{i j}=\#\{\text { arrows between vertices } i \text { and } j\}
$$

with the sign convention $b_{i j}>0$ if the arrows are oriented $i \rightarrow j$, and $b_{i j}<0$ for the opposite direction. Hence $B$ is a skew-symmetric matrix.

The $y$ seeds are $\left(\left(\underline{y}=\left(y_{1}, \ldots, y_{n}\right), \Gamma \equiv B\right)\right.$ where $\underline{y}$ is a set of variables and $B$ is the adjacency matrix of $\Gamma$.

The $y$-mutation at $k$ leads to a new seed $\left(\left(\underline{y^{\prime}}=\left(y_{1}^{\prime}, \ldots, y_{n}^{\prime}\right), \Gamma^{\prime} \equiv B^{\prime}\right)\right.$ given by $y_{k}^{\prime}=\frac{1}{y_{k}}$ and

$$
y_{j}^{\prime}= \begin{cases}y_{j}\left(1+y_{k}\right)^{b_{k j}} & \text { if } b_{k j} \geq 0 \\ y_{j}\left(1+y_{k}^{-1}\right)^{b_{k j}} & \text { if } b_{k j} \leq 0\end{cases}
$$

and $\Gamma^{\prime}=\mu_{k}(\Gamma)$ is the same as before.

If we define $y_{i}=\prod_{j} x_{j}^{b_{i j}}$ and $y_{i}^{\prime}=\prod_{j} x^{b_{i j}^{\prime}}{ }_{j}$ then the relationship between $x$ and $y$ mutations is such that the following diagram commutes:


Note that in the case when $B$ is not invertible the formulas $y_{i}=\prod_{j} x_{j}^{b_{i j}}$ lead to algebraic relations between the $y_{i}^{\prime} s$ and consequently in this case there is no expression for the $x$-variables in terms of the $y$-variables.
Exercise 2.6. Consider a quadrilateral with variables $a, b, c, d$ at its vertices. To the diagonal $\Delta=[a c]$ one assigns the variable $y_{\Delta}$ given by the cross ratio:

$$
y_{\Delta}=[a, b, c, d]=\frac{(a-d)(b-c)}{(a-b)(c-d)} .
$$

Show that the flip of the diagonal $\Delta_{1}$ (see figure below) leads to variables ( $y_{\Delta_{1}^{\prime}}, y_{\Delta_{2}^{\prime}}$ ) which are given in terms of $\left(y_{\Delta_{1}}, y_{\Delta_{2}}\right)$ by the same formula as the formula of a $y$-mutation at vertex 1 in the quiver $1 \longrightarrow 2$.


### 2.3 Compatible Poisson Bracket [10]

Recall a Poisson bracket is a map $\{-,-\}: A \times A \rightarrow A$, where $A$ is an algebra, such that $\{-,-\}$

1. is bilinear
2. is skew symmetric
3. satisfies the Jacobi identity

$$
\{\{a, b\}, c\}+\{\{c, a\}, b\}+\{\{b, c\}, a\}=0
$$

4. satisfies the Leibniz rule

$$
\{a b, c\}=a\{b, c\}+\{a, c\} b .
$$

Definition. A Poisson bracket on $A=\mathcal{A}(\Gamma)$ is called cluster compatible if for every cluster $\underline{x}$ of $A$ there exists a skew-symmetric matrix $\Omega^{\underline{x}}$ such that for each $x_{i}, x_{j} \in \underline{x}$ we have

$$
\left\{x_{i}, x_{j}\right\}=\Omega_{i j}^{\underline{x}} x_{i} x_{j}
$$

A Poisson bracket of this form is called log-canonical.

Theorem 2.7. If $(\underline{x}, B)$ is a seed of $\mathcal{A}(\Gamma)$ with $B$ invertible then the Poisson bracket defined by

$$
\left\{x_{i}, x_{j}\right\}:=c_{i j} x_{i} x_{j}
$$

where $C:=\left(c_{i j}\right)=B^{-1}$, is a cluster compatible Poisson bracket on $\mathcal{A}(\Gamma)$.
For non-invertible matrices $B$ it could be preferable to work with 2-forms instead of Poisson brackets.

Theorem 2.8. The 2 -form defined by

$$
\omega=\sum_{i<j} b_{i j} \frac{d x_{i} \wedge d x_{j}}{x_{i} x_{j}}
$$

is cluster compatible, i.e.

$$
\omega=\sum_{i<j} b_{i j}^{\prime} \frac{d x_{i}^{\prime} \wedge d x_{j}^{\prime}}{x_{i}^{\prime} x_{j}^{\prime}}
$$

for any seed ( $\underline{x}^{\prime}, B^{\prime}$ ).
When considering $y$-mutations the matrix $B$ provides with a good Poisson bracket in all cases, $B$ invertible or not.

Theorem 2.9. If $(\underline{y}, B)$ is a $y$-seed then

$$
\left\{y_{i}, y_{j}\right\}=b_{i j} y_{i} y_{j}
$$

is always cluster compatible, i.e. if $\left(\underline{y^{\prime}}, B^{\prime}\right)$ is another $y$-seed then

$$
\left\{y_{i}^{\prime}, y_{j}^{\prime}\right\}=b_{i j}^{\prime} y_{i}^{\prime} y_{j}^{\prime} .
$$

### 2.4 Pentagram map [11]

We consider the $y$-parameters of the twisted $n$-gons (see Problem 2 of the exercise sheet). They are related to the ( $a, b$ )-coordinates by

$$
\begin{aligned}
y_{2 i-1} & =-\left(a_{i} b_{i+2}\right)^{-1} \\
y_{2 i} & =-a_{i+2} b_{i+1}
\end{aligned}
$$

The $y$-parameters of the polygons change under the pentagram map according to the rule of mutation of $y$-seeds. More precisely we have the following description.

Theorem 2.10. Consider the bipartite quiver $\Gamma_{n}$ with vertices $\{1, \ldots, 2 n\}$ and arrows at each vertex $i$, with $i$ odd given by:


If $\underline{y}=\left(y_{1}, \ldots, y_{2 n}\right)$ are the $y$-parameters of a twisted $n$-gon $P$ and $\underline{y}^{\prime}=\left(y_{1}^{\prime}, \ldots, y_{2 n}^{\prime}\right)$ are the $y$ parameters of $T(P)$ then $\underline{y^{\prime}}=\mu_{\text {odd }}(\underline{y})$, where $\mu_{\text {odd }}$ is the composition of mutations at the odd vertices of $\Gamma_{n}{ }^{1}$.

Corollary 2.11. Let $B$ be the adjacency matrix of $\Gamma_{n}$. The Poisson bracket given by $\left\{y_{i}, y_{j}\right\}=b_{i j} y_{i} y_{j}$ is $T$-invariant.

[^0]The invariant Poisson bracket is the first ingredient one needs to establish the integrability of the map. The quiver $\Gamma_{n}$ also provides a combinatorial structure to get the conserved quantities (via perfect matchings on some lift of the quiver on the torus [7]).

All this gives a combinatorial setup to study the map $T$ and to conclude to the Liouville-Arnold integrability of the map acting on the space of twisted $n$-gons $\mathcal{P}_{n}$.

The integrability of $T$ acting on $\mathcal{P}_{n}$ and $\mathcal{C}_{n}$ was originally established in [21], [22] using a geometric approach. Find an analog of the above theorem for the action of the pentagram map on the closed $n$-gons is an open problem.
Remark. There are many recent developments around the pentagram map and its generalizations, [23], [24], [25], and around other similar cluster dynamics [26], [27] ...

### 2.5 Solutions

Solution of exercise 2.5.

1. One obtains the following sequence, where we can identify (up to relabeling) the first and last seeds:
$\binom{x_{1}, x_{2}}{1 \rightarrow 2} \xrightarrow{\mu_{1}}\binom{\frac{1+x_{2}}{x_{1}}, x_{2}}{1 \leftarrow 2} \xrightarrow{\mu_{2}}\binom{\frac{1+x_{2}}{x_{1}}, \frac{1+x_{1}+x_{2}}{x_{1} x_{2}}}{1 \rightarrow 2} \xrightarrow{\mu_{1}}\binom{\frac{1+x_{1}}{x_{2}}, \frac{1+x_{1}+x_{2}}{x_{1} x_{2}}}{1 \leftarrow 2} \xrightarrow{\mu_{2}}\binom{\frac{1+x_{1}}{x_{2}}, x_{1}}{1 \rightarrow 2} \xrightarrow{\mu_{1}}\binom{x_{2}, x_{1}}{1 \leftarrow 2}$
2. One gets exactly the five cluster variables computed in the previous question.

## 3 Lecture 3: Cluster Algebras and Cluster Dynamics (II)

### 3.1 Integrability of Somos-4 [12]

In lower dimensional cases, it can be possible to relax certain conditions required for the LiouvilleArnold integrability, and consider weaker versions of integrability. The Somos-4 recurrence provides an elegant example.

Recall the Somos-4 recurrence gives the map

$$
\varphi:\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right) \mapsto\left(\begin{array}{c}
x_{2} \\
x_{3} \\
x_{4} \\
\frac{x_{2} x_{4}+x_{3}^{2}}{x_{1}}
\end{array}\right)
$$

Proposition 3.1. Consider the Poisson bracket defined by $\left\{x_{i}, x_{j}\right\}:=(j-i) x_{i} x_{j}$ then

1. $\{-,-\}$ is $\varphi$-invariant
2. $y_{1}:=\frac{x_{1} x_{3}}{x_{2}}$ and $y_{2}=\frac{x_{2} x_{4}}{x_{3}^{2}}$ are both Casimirs
3. $F:=y_{1} y_{2}+y_{1}^{-1}+y_{2}^{-1}+\left(y_{1} y_{2}\right)^{-1}$ is $\varphi$-invariant
4. The Casimirs are not $\varphi$-invariant. One has

$$
\binom{y_{1}}{y_{2}} \xrightarrow{\varphi}\binom{y_{2}}{\frac{1+y_{2}}{y_{1} y_{2}^{2}}} .
$$

So we don't have Liouville-Arnold integrability, because of item 4 above.
The following set up allows us to view the Somos-4 recurrence as a cluster mutation. Consider the quiver

then $\varphi^{k}$ is given by $\mu_{k} \mu_{k-1} \ldots \mu_{1}$ (in the mutations the indices are taken modulo 4 ), see Problem 3 of the exercise sheet. The corresponding adjacency matrix

$$
B=\left(\begin{array}{cccc}
0 & 1 & -2 & 1 \\
-1 & 0 & 3 & -2 \\
2 & -3 & 0 & 1 \\
-1 & 2 & -1 & 0
\end{array}\right)
$$

is of rank 2 so unfortunately we have no useful Poisson bracket. Instead we will work with the $\varphi$-invariant 2 -form:

$$
\omega=\frac{d x_{1} \wedge d x_{2}}{x_{1} x_{2}}-2 \frac{d x_{1} \wedge d x_{3}}{x_{1} x_{3}}+\frac{d x_{1} \wedge d x_{4}}{x_{1} x_{4}}+3 \frac{d x_{2} \wedge d x_{3}}{x_{2} x_{3}}-2 \frac{d x_{2} \wedge d x_{4}}{x_{2} x_{4}}+\frac{d x_{3} \wedge d x_{4}}{x_{3} x_{4}} .
$$

The next theorem is stated for more general map $\varphi$ (corresponding to recurrence given by cluster mutation of a quiver which is mutation-periodic of period 1, see [12] and [9]).

Theorem 3.2. Let $\varphi: \mathbb{C}^{N} \rightarrow \mathbb{C}^{N}$ be given by mutation of a "good" quiver $Q=B$ with $\operatorname{rank}(B)=k$ then

1. There is a projection $\pi: \mathbb{C}^{N} \rightarrow \mathbb{C}^{k}$ and a symplectic map $\tilde{\varphi}: \mathbb{C}^{k} \rightarrow \mathbb{C}^{k}$ for a symplectic form $\tilde{\omega}$ satisfying $\pi^{*} \tilde{\omega}$ such that the following diagram commutes:

2. Let $v_{1}, \ldots, v_{k} \in \mathbb{Z}^{N}$ be a basis of $\operatorname{im}(B)$ then $\pi$ is defined by

$$
\left(x_{i}\right)_{i=1, \ldots, N} \rightarrow\left(y_{j}\right)_{j=1, \ldots, k}
$$

with $y_{j}=x_{1}^{v_{j 1}} x_{2}^{v_{j 2}} \ldots x_{n}^{v_{j n}}$
3. Let $v_{k+1} \ldots v_{N} \in \mathbb{Z}^{N}$ be a basis of $\operatorname{ker}(B)$ and form the $(N \times N)$-matrix $M$ whose rows contain all the vectors $v_{i}$, then $\tilde{\omega}$ is obtained as

$$
\tilde{\omega}=\sum \tilde{b}_{i j} \frac{d x_{i} \wedge d x_{j}}{x_{i} x_{j}}
$$

with $\tilde{B}=\left(\tilde{b}_{i j}\right)$ given by $\left(M^{-1}\right)^{T} B M^{-1}=\left(\begin{array}{cc}\tilde{B} & 0 \\ 0 & 0\end{array}\right)$.
When applying the above theorem to the Somos-4 recurrence map we can take $v_{1}=(1,-2,1,0)$ and $v_{2}=(0,1,-2,1)$ as a basis for $\operatorname{im}(B)$ and $v_{3}=(1,1,1,1)$ and $v_{4}=(1,2,3,4)$ as a basis for $\operatorname{ker}(B)$. The commutative diagram is

and the 2 -form

$$
\tilde{\omega}=\frac{d y_{1} \wedge d y_{2}}{y_{1} y_{2}}
$$

gives a non-degenerate, $\tilde{\varphi}$ invariant Poisson bracket $\left\{y_{1}, y_{2}\right\}=y_{1} y_{2}$. We also have that

$$
F=y_{1} y_{2}+y_{1}^{-1}+y_{2}^{-1}+\left(y_{1} y_{2}\right)^{-1}
$$

is $\tilde{\varphi}$-invariant, hence $\tilde{\varphi}$ is Liouville-Arnold integrable.

Remark. We informally recall some of the properties of 2-forms. The space of 1-forms $E$ is a $K(M)$ vector space with basis $\left\{e_{i}=\frac{d x_{i}}{x_{i}}\right\}$ and its dual space $E^{*}$ is the space of vector fields with dual basis $\left\{e_{i}^{*}=x_{i} \frac{\partial}{\partial x_{i}}\right\}$ so $\left\langle e_{i}, e_{j}^{*}\right\rangle=\delta_{i j}$. The space of 2-forms $\Lambda^{2} E$ has a basis

$$
\left\{\frac{d x_{1} \wedge d x_{j}}{x_{i} x_{j}}: i<j\right\}
$$

and the dual space of bivector fields $\Lambda^{2} E^{*}$ has a basis

$$
\left\{x_{i} x_{j} \frac{\partial}{\partial x_{i}} \wedge \frac{\partial}{\partial x_{j}}: i<j\right\} .
$$

An $\omega \in \Lambda^{2} E$ gives a skew bilinear form on $E^{*}$, given by $\omega(-,-)$. As usual one defines

$$
\operatorname{ker} \omega:=\left\{u \in E^{*}: \omega(u, v)=0, \quad \forall v \in E^{*}\right\} .
$$

Any $F \in K(M)$ gives a 1-form

$$
d F:=\sum_{i} \frac{\partial f}{\partial x_{i}} d x_{i}
$$

and $u \wedge v \in \Lambda^{2} E^{*}$ gives a Poisson bracket on $K(M)$ :

$$
\{F, G\}=\langle d F \wedge d G, u \wedge v\rangle=\langle d F, u\rangle\langle d G, v\rangle-\langle d F, v\rangle\langle d G, u\rangle
$$

When $\omega=\sum_{i<j} b_{i j} \frac{d x_{i} \wedge d x_{j}}{x_{i} x_{j}}$ is non-degenerate, the natural dual object in $\Lambda^{2} E^{*}$ is the bivector $W=$ $\sum_{i<j} c_{i j} x_{i} x_{j} \frac{\partial}{\partial x_{i}} \wedge \frac{\partial}{\partial x_{j}}$ with $\left(c_{i j}\right)=\left(b_{i j}\right)^{-1}$.
Remark. 1. The following are a basis for $\operatorname{ker}(\omega)$

$$
v_{1}=x_{1} \frac{\partial}{\partial x_{1}}+x_{2} \frac{\partial}{\partial x_{2}}+x_{3} \frac{\partial}{\partial x_{3}}+x_{4} \frac{\partial}{\partial x_{4}}, \quad v_{2}=x_{1} \frac{\partial}{\partial x_{1}}+2 x_{2} \frac{\partial}{\partial x_{2}}+3 x_{3} \frac{\partial}{\partial x_{3}}+4 x_{4} \frac{\partial}{\partial x_{4}} .
$$

2. $\left\langle v_{i}, y_{j}\right\rangle=0$ for $i, j \in\{1,2\}$
3. The Poisson bracket $\{F, G\}=<d F \wedge d G, v_{1} \wedge v_{2}>$ gives the bracket $\left\{x_{i}, x_{j}\right\}=(j-i) x_{i} x_{j}$ of Proposition 3.1 which has obviously $y_{1}$ and $y_{2}$ as Casimirs, but which is, for non obvious reasons, $\{-,-\} \varphi$-invariant.
4. Computing $\tilde{\omega}=\frac{d y_{1} \wedge d y_{2}}{y_{1} y_{2}}$ in terms of $x_{i}^{\prime} s$ gives back $\omega$.

### 3.2 T-systems

Recall the octahedron recurrence

$$
T_{i-1, j, k} T_{i+1, j, k}-T_{i, j-1, k} T_{i, j+1, k}=T_{i, j, k-1} T_{i, j, k+1}
$$

Assume that $i+j+k$ is odd. We restrict the situation by assuming that we get zeroes outside of an infinite parallelepiped bordered by ones, see the conditions (3).


The system is the map

$$
\varphi: \mathbb{C}^{r w} \rightarrow \mathbb{C}^{r w} \quad \varphi\left(x_{j, k}^{0}\right)=\left(x_{j, k}^{1}\right)
$$

where $\left(x_{j, k}^{0}\right)$ are free variables located on a "transversed slice" in the parallelepiped (the slice has an accordion shape as we have removed the points with $i+j+k$ even) and ( $x_{j, k}^{1}$ ) the variables on the next slice. Consider the bipartite quiver of type $\mathbf{A}_{r} \times \mathbf{A}_{w}$ :


and the sequences of mutations:

$$
\mu_{\circ}=\prod_{k \text { white }} \mu_{k} \quad \mu_{\bullet}=\prod_{k \text { black }} \mu_{k}
$$

Theorem 3.3. $\varphi$ is given by $\mu=\mu_{\bullet} \mu_{\circ}$
Theorem 3.4. $\varphi$ is $w+r+2$ periodic. ("Zamolodchikov periodicity" [13])
Remark. This cluster structure is associated to $G r_{r+1, w+r+2}$, [28].

## 4 Lecture 4: Coxeter's friezes and first generalizations [14]

### 4.1 Definition and some nice properties [15] [16]

Coxeter's frieze patterns are arrays of numbers satisfying the following properties:
(i) the array has finitely many rows, all of them being infinite on the right and left,
(ii) the first two top rows are a row of 0 's followed by a row of 1 's, and the last two bottom rows are a row 1 's followed by a row of 0 ' ${ }^{2}$,
(iii) consecutive rows are displayed with a shift, and every four adjacent entries $a, b, c, d$ forming a diamond

$$
a \begin{array}{cc} 
& b \\
& \\
& d
\end{array}
$$

satisfy the unimodular rule: $a d-b c=1$.
The number of rows strictly between the border rows of 1's is called the width of the frieze (we will use the letter $m$ for the width). The following array (5) is an example of a frieze pattern of width $m=4$, containing only positive integer numbers.

| row 0 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | $\ldots$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| row 1 | $\ldots$ |  | 4 |  | 2 |  | 1 |  | 3 |  | 2 |  | 2 |  | 1 |  |
| row 2 |  | 3 |  | 7 |  | 1 |  | 2 |  | 5 |  | 3 |  | 1 |  | $\ldots$ |
| $\ldots$ | $\ldots$ |  | 5 |  | 3 |  | 1 |  | 3 |  | 7 |  | 1 |  | 2 |  |
| row $w$ |  | 3 |  | 2 |  | 2 |  | 1 |  | 4 |  | 2 |  | 1 |  | $\ldots$ |
| row $w+1$ | $\ldots$ |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  |

[^1]The definition allows the frieze to take its values in any ring with unit. Coxeter studies the properties of friezes with entries that are positive real numbers (apart from the border rows of 0 's), and with a special interest in the case of positive integers.

The condition of positivity is quite strong but guarantees a certain genericity of the frieze. We will work with a less restrictive condition. We will consider friezes with real or complex entries, and we will assume that they satisfy the following extra condition:
(iv) every adjacent $3 \times 3$-submatrix in the array has determinant 0 .

Friezes satisfying the condition (iv) are called tame. Coxeter's friezes with no zero entries (in particular friezes with positive numbers) are all tame. The statements established in [15] for the friezes with positive entries still hold for tame friezes (the proofs can be easily adapted).

Proposition 4.1. Properties of tame friezes.

1. Rows in a frieze of width $w$ are periodic with period dividing $w+3$.
2. Friezes are invariant under a glide reflection with respect to the horizontal median line of the pattern.
3. If $a_{1}, a_{2}, \ldots, a_{n},(n=w+3)$, are the entries in the first row, then all the entries in the frieze can be expressed as polynomials in $a_{i}^{\prime} s$ with integer coefficients.
4. If $x_{1}, x_{2}, \ldots, x_{w}$ are the entries forming a zig-zag from top to bottom in the frieze, then all the entries in the frieze can be expressed as Laurent polynomials in $x_{i}^{\prime} s$ with positive integer coefficients.

Extending the pattern (5) one observes the property of glide reflection:


## Friezes with positive integers.

If we are interested in building friezes with positive integers, the above properties 3 and 4 may help but are not fully satisfying.

The property 3 ensures that if the first row of a given frieze consists of positive integers then all the rest of the frieze consists of positive integers. But starting with arbitrary positive integer values on the first row does not necessarily lead to a "closed frieze" (i.e. one can compute the rest of the frieze with the unimodular rule but there is no guarantee to end up with a row of 1's after $w$ steps).

From property 4, one can easily obtain a closed frieze of positive integers by placing a zig-zag of 1 's in the fireze and computing the rest by using the unimodular rule. But not all friezes with positive integers contain a zig-zag of 1's, for instance

|  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | $\cdots$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\ldots$ |  | 1 |  | 3 |  | 1 |  | 3 |  | 1 |  | 3 |  |
|  | 2 |  | 2 |  | 2 |  | 2 |  | 2 |  | 2 |  | $\ldots$ |
| $\ldots$ |  | 3 |  | 1 |  | 3 |  | 1 |  | 3 |  | 1 |  |
|  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | $\cdots$ |

Here is the complete classification of frieze with positive integers due to John Conway.

Theorem 4.2 (Conway's correspondence [16]). Frieze patterns of width $w=n-3$ with positive integers are in one-to-one correspondence with the triangulations of a convex n-gon. If $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ is the cycle on the first row of the frieze, then $a_{i}$ is the number of triangles adjacent to the $i$-th vertex in the corresponding triangulated $n$-gon.
Example 4.3. The first row of the frieze (5) is the cycle (4, 2, 1, 3, 2, 2, 1) and this corresponds to the following triangulated heptagon.


See Problem 4 and 5 on the exercise sheet for more properties of the friezes.
Other surprizing links between friezes and classical objects (Fibonacci sequence, Chebyshev polynomials, Farey sequences, continued fractions, cross ratios...) can be found in [15, 16].

### 4.2 The variant of 2-friezes [17]

The variant of 2-friezes consist in arrays of numbers in the plane satisfying the following conditions:
(i') the array has finitely many rows, all of them being infinite on the right and left,
(ii') the first three top rows are two rows of 0 's followed by a row of 1 's, and the last two bottom rows are a row 1 's followed by two rows of 0 's,
(iii') every five adjacent entries $a, b, c, d, e$ forming a diamond centered at $e$

$$
\begin{array}{lll}
* & b & * \\
a & e & d \\
* & c & *
\end{array}
$$

satisfy $a d-b c=e$.
The number of rows strictly between the bordering rows of 1's is called the width of the frieze and denoted by $w$.

Example 4.4. The following array is an example of 2-frieze of width 4.

$$
\begin{array}{lccccccccccccccccc}
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\ldots & 3 & 7 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 5 & 10 & 3 & 1 & 2 & 3 & 2 & \cdots \\
\cdots & 11 & 5 & 10 & 6 & 2 & 2 & 2 & 2 & 8 & 15 & 5 & 7 & 5 & 1 & 1 & 7 & \cdots \\
\cdots & 8 & 15 & 5 & 7 & 5 & 1 & 1 & 7 & 11 & 5 & 10 & 6 & 2 & 2 & 2 & 2 & \cdots  \tag{6}\\
\cdots & 2 & 5 & 10 & 3 & 1 & 2 & 3 & 2 & 3 & 7 & 4 & 2 & 2 & 2 & 2 & 2 & \cdots \\
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots
\end{array}
$$

Note that Coxeter's friezes are not a priori particular cases of 2-friezes: adding 1's at the centers of the diamond in a Coxeter frieze would lead to diamonds of 1's for which the rule $a d-b c=e$ is not satisfied.

But the 2 - friezes enjoy similar properties.
Proposition 4.5. Properties of generic ${ }^{3}$ 2-friezes.

1. Rows in a 2-frieze of width $w$ are periodic with period dividing $2(w+4)$.
2. Friezes are invariant under a glide reflection with respect to the horizontal median line of the pattern.

[^2]3. If $a_{1}, b_{1}, a_{2}, b_{2} \ldots, a_{n}, b_{n}(n=w+4)$, are the entries in the first row, then all the entries in the frieze can be expressed as polynomials in $a_{i}, b_{i}^{\prime} s$ with integer coefficients.
4. If $x_{1}, x_{2}, \ldots, x_{w}, y_{1}, y_{2}, \ldots, y_{w}$ are the entries forming a double zig-zag from top to bottom in the frieze, then all the entries in the frieze can be expressed as Laurent polynomials in $x_{i}, y_{i}^{\prime} s$ with positive integer coefficients.

Remark. What is called a double zig-zag? The entry $x_{i+1}$ should be immediately at the right or left under $x_{i}$ (and similarly for $y_{i+1}$ and $y_{i}$ ) and $x_{i}, y_{i}$ should touch each other. Possible configurations (up to switch of $x$ 's and $y$ 's) for a double zig-zag path are


The glide reflection for the 2-frieze (6) can be observed on the following picture:

The glide reflection and the Laurent phenomenon can be observed in the following example of 2-frieze:

$$
\begin{array}{cccccccccc}
y_{2} \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & x_{1} & y_{1} & \frac{y_{1}+x_{2}}{x_{1}} & \frac{\left(y_{1}+x_{2}\right)\left(y_{2}+x_{1}\right)}{x_{1} y_{1} y_{2}} & \frac{\left(x_{1}+y_{2}\right)\left(x_{2}+y_{1}\right)}{x_{2} y_{1} y_{2}} & \frac{x_{1}+y_{2}}{x_{2}} & y_{2} & x_{2} & \cdots \\
\cdots & y_{2} & x_{2} & \frac{x_{2}+y_{1}}{y_{2}} & \frac{\left(x_{2}+y_{1}\right)\left(x_{1}+y_{2}\right)}{y_{2} x_{2} x_{1}} & \frac{\left(y_{2}+x_{1}\right)\left(y_{1}+x_{2}\right)}{y_{1} x_{1} x_{2}} & \frac{y_{2}+x_{1}}{y_{1}} & x_{1} & y_{1} & \cdots \\
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots
\end{array}
$$

## More properties.

The 2 friezes have a geometric interpretation.
Theorem 4.6. The $2 n$-tuple $\left(a_{i}, b_{i}\right) \in \mathbb{C}^{2 n}$ is the first row of a 2-frieze if and only if $\left(a_{i}, b_{i}\right)$ are the ( $a, b$ )-coordinates of a closed $n$-gon in $\mathcal{P}_{n}$.

In other words the space of 2 -friezes is identified with the space $\mathcal{C}_{n}$ of closed $n$-gons.
Example 4.7. The 2 -frieze of width 1 is 10 periodic. Its non trivial row $b_{1}, a_{1}, \ldots, b_{5}, a_{5}$ can be easily computed using the frieze rule. One gets:

$$
\begin{array}{ccccccccc}
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & x & y & \frac{y+1}{x} & \frac{x+y+1}{x y} & \frac{x+1}{y} & x & y & \cdots \\
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots
\end{array}
$$

that give the same solution as in Exercise 1.3.

## 2-frieze of positive integers.

There is no analog of Conway's correspondence. One only knows how many 2-friezes with positive integers do exist for a given width.

| width | \# of friezes | status |
| :---: | :---: | :--- |
| 1 | 5 | proved |
| 2 | 51 | in [17] |
| 3 | 868 | conjectured |
| 4 | 26952 | in [29] |
| $>4$ | $\infty$ | proved in [30] |

## $4.3 \quad S L_{r+1}$-friezes [18]

$S L_{r+1}$-friezes are natural generalizations of Coxeter's friezes, the latter corresponding to the case $r=1$. An $S L_{r+1}$-frieze is an array of numbers in the plane such that
(i") the array has finitely many rows, all of them being infinite on the right and left,
(ii") the first $r+1$ top rows are $r$ rows of 0 's followed by a row of 1 's, and the last $r+1$ bottom rows are a row 1 's followed by $r$ rows of 0 's,
(iii") consecutive rows are displayed with a shift, and every adjacent entries in a diamond of size $(r+1) \times(r+1)$

form a matrix of determinant 1 .
We will also add the condition of genericity for tame friezes:
(iv") every adjacent $(r+2) \times(r+2)$-submatrix in the array has determinant 0 .
The number of rows strictly between the border rows of 1's is again called the width of the frieze.
The arrays of Figure 2 show two $S L_{3}$-friezes of width 4 .

| $\cdots$ | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | $\cdots$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\cdots$ |  | 7 |  | 2 |  | 2 |  | 2 |  | 5 |  | 3 |  | 2 |  | 2 | $\cdots$ |
| $\cdots$ | 11 |  | 10 |  | 2 |  | 2 |  | 8 |  | 5 |  | 5 |  | 1 |  | $\cdots$ |
| $\cdots$ |  | 15 |  | 7 |  | 1 |  | 7 |  | 5 |  | 6 |  | 2 |  | 2 | $\cdots$ |
| $\cdots$ | 2 |  | 10 |  | 1 |  | 3 |  | 3 |  | 4 |  | 2 |  | 2 |  | $\cdots$ |
| $\cdots$ |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | $\cdots$ |  |

$\left.\begin{array}{llllllllllllllllll}\cdots & & 1 & & 1 & & 1 & & 1 & & & 1 & & 1 & & 1 & & 1\end{array}\right)$

Figure 2: $S L_{3}$-friezes of width 4

## Link with $T$-systems.

Let $F$ be a tame $S L_{r+1}$ frieze, and denote by $\left(a_{i, j}\right)_{i \in \mathbb{Z},-r-1 \leq j \leq w+r+1}$ its entries (including the rows of 0 's and 1 's). Define the adjacent minors of $F$ of order $\ell+1$ based on $(i, j)$ as

$$
A_{i, j}^{(\ell+1)}=\operatorname{det}\left(\begin{array}{llll}
a_{i, j} & a_{i, j+1} & \ldots & a_{i, j+r \ell}  \tag{7}\\
a_{i+1, j} & a_{i+1, j+1} & \ldots & a_{i+1, j+\ell} \\
\ldots & \ldots & & \ldots \\
a_{i+\ell, j} & a_{i+\ell, j+1} & \ldots & a_{i+\ell, j+\ell}
\end{array}\right) .
$$

and the $\ell$-derived array of $F$ as

$$
\partial_{\ell} F:=\left(A_{i j}^{(\ell)}\right) .
$$

Consider the following construction. Embed the frieze $F$ in the discrete $3 D$-space, placing the array in a horizontal plane at height 1 . In the horizontal planes just above, place at height $\ell$ the array $\partial_{\ell} F$ of the $\ell$-minors of $F$. At height $\ell=r+1$ the array consists only of 1 's due to the frieze rule. And at height $\ell \geq r+2$ the array consists only of 0 's due to the tameness of the frieze $F$.


Proposition 4.8. The above construction leads to the following.

1. The superposition of the planes satisfies the equation of a T-system with restriction as in ??, and conversely all restricted $T$-systems are obtained from the derived arrays of an $S L_{r+1}$-frieze.
2. The last non-trivial array, i.e. the array at height $\ell=r$, forms an $S L_{r+1}$-frieze denoted by $F^{*}$.
3. The first vertical plane above $F$ containing non-0's and no-1's values forms an array which is an $S L_{w+1}$-frieze of width $r$ denoted by $F^{\mathcal{G}}$.
4. The next vertical planes consist in the successive derived arrays of $F^{\mathcal{G}}$.

Corollary 4.9. $S L_{r+1}$-friezes of width $w$ are $(r+w+2)$-periodic.
Remark. The space of $S L_{r+1}$-friezes of width $w$ can be identified with the space of $n$-gons in the projective space $\mathbb{P}^{r}$ (with $\left.n=r+w+2\right)$. Under this identification $F^{*}$ corresponds to the "projective dual" of $F$ and $F^{\mathcal{G}}$ corresponds to the "Gale dual" of $F$. See [31], for more details.

Remark. In the case where $r=2$, the above construction leads to only two superposed arrays $F$ and $F^{*}$. When projecting the two arrays in the same plane one obtains a 2-frieze. The arrays of Figure 2 lead to the 2-frieze (6).

### 4.4 More variants

Let us mention other natural generalizations of Coxeter's friezes that have been studied recently (see [14, §5] for many other variants).

- remove the bordering rows of 0 's and 1 's $\rightarrow$ " $S L_{2}$-tilings of the plane", [32];
- remove only the bottom bordering rows of 0 's and 1 's $\rightarrow$ "infinite friezes", [33], [34], [35];
- remove the tameness condition $\rightarrow$ "wild friezes", [36];
- use Conway's correspondence with $d$-angulations, [37]

In the next section, we will study a generalization of Coxeter's friezes using quiver representations.

## 5 Lecture 5: Friezes and representation theory [14]

### 5.1 Friezes defined over a quiver [19]

Let $\mathcal{Q}$ be a connected acyclic quiver. The set of vertices $\mathcal{Q}_{0}$ and the set of arrows $\mathcal{Q}_{1}$ are assumed to be finite. We denote by $n$ the cardinality of $\mathcal{Q}_{0}$ and often identify this set with the elements $\{1,2, \ldots, n\}$.

The repetition quiver $\mathbb{N} \mathcal{Q}$ is defined by:

- the vertices of $\mathbb{N} \mathcal{Q}$ are the couples $(m, i), m \in \mathbb{N}, i \in \mathcal{Q}_{0}$,
- the arrows are

$$
(m, i) \longrightarrow(m, j) \quad \text { and } \quad(m, j) \longrightarrow(m+1, i),
$$

for all $m \in \mathbb{N}$, whenever $i \longrightarrow j$ is an arrow in $\mathcal{Q}_{1}$.
We denote by $\tau$ the translation on the vertices of $\mathbb{N} \mathcal{Q}$ defined by

$$
\begin{equation*}
\tau:(m, i) \mapsto(m-1, i) \tag{8}
\end{equation*}
$$

Example 5.1. We give examples of repetition quivers for Dynkin quivers over $\mathbf{A}, \mathbf{D}, \mathbf{E}$ with linear orientations (other orientations will lead to other shape of friezes):

1) Case $\mathcal{Q}=\mathbf{A}_{n}$ :

2) Case $\mathcal{Q}=\mathbf{D}_{n}$ :

3) Case $\mathcal{Q}=\mathbf{E}_{n}, n=6,7,8$.


Definition. (multiplicative and additive frieze over $\mathcal{Q}$ )

1. A frieze over $\mathcal{Q}$ is a function on the repetition quiver

$$
f: \mathbb{N Q} \rightarrow \mathcal{A},
$$

assigning at each vertex of $\mathbb{N} \mathcal{Q}$ an element in a fixed commutative ring with unit $\mathcal{A}$, so that the assigned values satisfy some "mesh relations" read out of the oriented graph $\mathbb{N Q}$.
2. The function $f$ will be called an additive frieze if it satisfies for all $v \in \mathbb{N} \mathcal{Q}_{0}$,

$$
f(\tau v)+f(v)=\sum_{\substack{\alpha \in \mathbb{N} \mathcal{Q}_{1}: \\ w \stackrel{\alpha}{\longrightarrow} v}} f(w)
$$

3. The function $f$ will be called a multiplicative frieze if it satisfies for all $v \in \mathbb{N} \mathcal{Q}_{0}$,

$$
f(\tau v) f(v)=1+\prod_{\substack{\alpha \in \mathbb{N} \mathcal{Q}_{1}: \\ w \xrightarrow{\alpha} v}} f(w) .
$$

Remark. Coxeter's friezes correspond to friezes over A. The first variant of friezes using quivers of type $\mathbf{D}$ was introduced in [38] and then generalized in [19]. In [19], they also define friezes in a more general way using Cartan matrices (or valued quivers). If $C=\left(c_{i j}\right)_{1 \leq i, j \leq n}$ is a Cartan matrix, the associated frieze is defined on $\mathbb{N} \times\{1, \ldots, n\}$ by

$$
f(m, j) f(m+1, j)=1+\prod_{c_{i j} \neq 0} f(m+1, i)^{-c_{i j}} \prod_{c_{j k} \neq 0} f(m, k)^{-c_{j k}} .
$$

In the simply laced case, i.e. in type $\mathbf{A D E}$, the friezes obtained using the Cartan matrices are the same as the friezes over the quiver. Be aware that the frieze over the Kronecker quiver is not the same as a frieze of Cartan type $\mathbf{B}_{2}$ or $\mathbf{C}_{2}$.

Remark. Additive friezes are classical objects in Auslander-Reiten theory, more often called "additive functions", see e.g. [39] and references therein. Multiplicative friezes naturally appear in [40]. Other natural rules for friezes appear in the context of cluster algebras. For instance, cluster-additive friezes and tropical friezes with recurrence rules

$$
f(\tau v)+f(v)=\sum_{w \xrightarrow{\alpha} v} \max (f(w), 0), \quad f(\tau v)+f(v)=\max \left(\sum_{w \xrightarrow{\alpha} v} f(w), 0\right),
$$

respectively, are introduced and studied in [41], and [42].
Example 5.2. (1) A multiplicative frieze over $\mathbf{D}_{5}$ (computed in [38]):

(2) An additive frieze over $\mathbf{D}_{5}$ :

(3) A multiplicative frieze over the Kronecker quiver:

(4) An additive frieze over the Kronecker quiver:


As in the Coxeter case we will need a notion of genericity of the friezes. We define formal generic friezes.

Definition. Let us fix a set of indeterminates $\left\{x_{1}, \ldots, x_{n}\right\}$. The generic additive and multiplicative friezes, denoted by $f_{a d}$ and $f_{m u}$ respectively, are defined by assigning the value $x_{i}$ to the vertex $(0, i)$ for all $1 \leq i \leq n$. One gets

$$
f_{a d}: \mathbb{N} \mathcal{Q} \rightarrow \mathbb{Z}\left[x_{1}, \ldots, x_{n}\right], \quad f_{m u}: \mathbb{N} \mathcal{Q} \rightarrow \mathbb{Q}\left(x_{1}, \ldots, x_{n}\right)
$$

We will refer to $x_{i}$ 's as the initial values of the friezes.
Remark. If $x_{1}, \ldots, x_{n}$ are not indeterminates but some given values in a ring $\mathcal{A}$, one may find different multiplicative friezes with same initial values $x_{i}$ 's. Indeed, it may happen that $f(\tau v)=0$ for some $v$, and thus the multiplicative rule does not allow us to define uniquely $f(v)$. Below, we give an example of two different multiplicative friezes on the repetition quiver of $\mathbf{A}_{3}$ with same initial values $(0,-1,0)$.


## Symmetry of friezes.

A frieze $f: \mathbb{N} \mathcal{Q} \rightarrow \mathcal{A}$ is periodic, if there exists an integer $N \geq 1$ such that $f \tau^{-N}=f$. The following theorem is a consequence in terms of friezes of classical results from the theory of quiver representations and the theory of cluster algebras, see [14] for a proof.

Theorem 5.3. The friezes $f_{a d}$ and $f_{m u}$ over a quiver $\mathcal{Q}$ are periodic if and only if $\mathcal{Q}$ is a Dynkin quiver of type $\mathbf{A}_{n}, \mathbf{D}_{n}$ or $\mathbf{E}_{6,7,8}$; in these cases the periods ${ }^{4}$ are

| periods | $f_{a d}$ | $f_{m u}$ |
| :--- | :---: | :---: |
| $\mathbf{A}_{n}$ | $n+1$ | $n+3$ |
| $\mathbf{D}_{n}$ | $2(n-1)$ | $2 n$ |
| $\mathbf{E}_{6,7,8}$ | $12,18,30$ | $14,20,32$ |.

Remark. As a corollary, all additive friezes are periodic (as they are all coming form evaluations of $f_{a d}$ ). But this is not the case for multiplicative friezes, see (9) for an example of non-periodic multiplicative frieze.

### 5.2 Friezes and quiver representations [20]

Let $\mathcal{Q}$ be a finite acyclic connected quiver, and $\mathcal{Q}^{o p}$ the same quiver with reversed orientation. We consider the category $\operatorname{rep}_{\mathbb{C}}\left(\mathcal{Q}^{o p}\right)$ of representations of $\mathcal{Q}^{o p}$. Recall that a representation of $\mathcal{Q}^{o p}$ is a collection of spaces and maps

- $\left(M_{i}\right)_{i \in \mathcal{Q}_{0}}$, where $M_{i}$ is a $\mathbb{C}$-vector space attached to the vertex $i$ of $\mathcal{Q}^{o p}$,
- $\left(f_{\alpha}: M_{i} \rightarrow M_{j}\right)_{i \rightarrow j}$, where $f_{\alpha}$ is a $\mathbb{C}$-linear map attached to the arrow $i \xrightarrow{\alpha} j$ of $\mathcal{Q}^{o p}$.

There are natural notions of subrepresentations, direct sums, morphisms of representations, indecomposable representations, ...

The Auslander-Reiten quiver (AR quiver) of $\operatorname{rep}_{\mathbb{C}}\left(\mathcal{Q}^{o p}\right)$ is the quiver $\Gamma_{\mathcal{Q}^{o p}}$ defined by:

- vertices: isomorphism classes of indecomposable objects [M],
- arrows: $[M] \xrightarrow{\ell}[N]$, if the space of irreducible morphisms from $M$ to $N$ is of dimension $\ell$.

The irreducible morphisms are those that are not compositions, or combinations of compositions, of other non-trivial morphisms. In other words the AR quiver gives the elementary bricks (representations and morphisms) to construct rep ${ }_{\mathbb{C}}\left(\mathcal{Q}^{o p}\right)$.

The following theorem collect classical results relating the AR quiver $\Gamma_{\mathcal{Q}^{o p}}$ (or part of it) to the repetition quiver over $\mathcal{Q}$.

Theorem 5.4. Let $\mathcal{Q}$ be a finite acyclic connected quiver.

1. The projective modules all belong to the same connected component $\Pi_{\mathcal{Q}^{o p}}$ of $\Gamma_{\mathcal{Q}^{o p}}$.
2. In the case when $\mathcal{Q}$ is a Dynkin quiver of type $\mathbf{A}, \mathbf{D}, \mathbf{E}$, one has $\Pi_{\mathcal{Q}^{o p}} \simeq \Gamma_{\mathcal{Q}^{o p}}$ and they can be embedded as a finite full subquiver of the repetition quiver:

$$
\Pi_{\mathcal{Q}^{o p}} \simeq \Gamma_{\mathcal{Q}^{o p}} \leftrightarrow \mathbb{N Q}
$$

3. In all other cases, $\Gamma_{\mathcal{Q}^{o p}}$ is not connected. The component $\Pi_{\mathcal{Q}^{o p}}$ is isomorphic to the full repetition quiver:

$$
\Pi_{\mathcal{Q}^{o p}} \xrightarrow{\sim} \mathbb{N} \mathcal{Q}^{\mathrm{op}} .
$$

4. For all $i \in \mathcal{Q}_{0}$, the standard projective module $P_{i}$ in $\operatorname{rep}_{\mathbb{C}}\left(\mathcal{Q}^{o p}\right)$ identifies with the vertices $(0, i)$, in $\mathbb{N} Q^{\text {op }}$.

[^3]5. For $N, M, E_{i}$ indecomposable representations, one has

6. The AR translation and the map $\tau$ of (8) coincide.

## Generalized friezes from the AR quiver.

From the above theorem one can now see friezes as functions on the AR quiver.
There are natural additive friezes given by the dimension vectors.

$$
\underline{\operatorname{dim}}: \Gamma_{\mathcal{Q}^{o p}} \longrightarrow \mathbb{Z}^{n} \text { gives an additive frieze from } \mathbb{N Q} \text { to } \mathbb{Z}^{n} .
$$

By projection on the $i$-th component one also gets additive friezes $d_{i}:=p r_{i} \circ \underline{\operatorname{dim}}: \mathbb{N} \mathcal{Q} \rightarrow \mathbb{Z}$. The $d_{i}^{\prime} s$ form a $\mathbb{Z}$-basis for the $\mathbb{Z}$-module of additive friezes.

There are "natural" multiplicative friezes given by the Caldero-Chapoton map (CC map). For $M$ an indecomposable representation of $\mathcal{Q}^{o p}$ define

$$
x_{M}:=C C(M)=\frac{1}{x_{1}^{d_{1}} x_{2}^{d_{2}} \ldots x_{n}^{d_{n}}} \sum_{e \in \mathbb{N}_{0}} \chi\left(\operatorname{Gr}_{e}(M)\right) \prod_{i \in \mathcal{Q}_{0}} x_{i}^{\sum_{j \rightarrow i} e_{j}+\sum_{i \rightarrow j}\left(d_{j}-e_{j}\right)},
$$

where

- $\left(d_{1}, d_{2}, \ldots, d_{n}\right)=\underline{\operatorname{dim}} M$,
- $\operatorname{Gr}_{e}(M):=\{N$ subrepresentations of $M$ s.t. $\underline{\operatorname{dim}} N=e\}$ is the quiver Grassmannian,
- $\chi$ is the Euler characteristic.

The CC map and the following theorem was first established in type $\mathbf{A}$ in [40] and have been extended to other type later.

Theorem 5.5. For the situation (10) as above, one has

$$
x_{M} x_{N}=1+\prod_{i} x_{E_{i}}
$$

Moreover the $x_{M}$ 's are cluster variables in the cluster algebra $\mathcal{A}_{\mathbb{Q}}(\mathcal{Q})$ generated from the initial seed $\left(\left(x_{1}, \ldots, x_{n}\right), \mathcal{Q}\right)$.

Corollary 5.6. The CC map defines a multiplicative frieze which is the same as $f_{m u}$, and the entries in the frieze $f_{m u}$ are all cluster variables of $\mathcal{A}_{\mathbb{Q}}(\mathcal{Q})$

## Generalized friezes with positive integer entries.

There is no analog of Conway's correspondence for the generalized friezes using quivers or Cartan matrices. One has information on the number of such friezes:

| type | \# of friezes | status |
| :---: | :---: | :---: |
| $\mathbf{A}_{n}$ | Catalan | proved in [16] |
| $\mathbf{B}_{n}$ | $\sum_{m \leq \sqrt{n+1}}\binom{2 n-m^{2}+1}{n}$ | proved in [29] |
| $\mathbf{C}_{n}$ | $\binom{2 n}{n}$ | proved in [29] |
| $\mathbf{D}_{n}$ | $\sum_{m=1}^{n} \#\{$ divisors of $m\}\binom{2 n-m-1}{n-m}$ | proved in [29] |
| $\mathbf{E}_{6,7,8}$ | $868,4400,26952$ | conjectured in [29] |
| $\mathbf{F}_{4}$ | 112 | conjectured in [29] |
| $\mathbf{G}_{2}$ | 9 | proved in [29] |
| otherwise | $\infty$ | proved in [30] |

In the above table the Dynkin types refer to the Cartan matrices used to build the friezes. In type ADE these friezes correspond to the friezes over the associated quiver as defined in $\S 5.1$, (see also the first Remark in §5.1). Friezes of type $\mathbf{D}_{4}, \mathbf{E}_{6}, \mathbf{E}_{8}$ also correspond to $S L_{3}$-friezes of width 2, 3, 4, respectively.

### 5.3 Some open problems

- Prove (or disprove) the conjectured numbers in the above table.
- Find analogues of Conway's correspondence (Thm 4.2) for generalized friezes.
- Find combinatorial interpretations of the entries for the positive integer valued friezes.
- There is an easy way to get a generalized frieze with positive integer entries:

$$
\left\{\text { clusters of } \mathcal{A}_{\mathbb{Q}}(\mathcal{Q})\right\} \rightarrow\{\text { friezes of type } \mathbb{Q} \text { with positive integer entries }\}
$$

The map is defined as follows. Choose any cluster $\left(u_{1}, \ldots, u_{n}\right)$ in $\mathcal{A}_{\mathbb{Q}}(\mathcal{Q})$. All the cluster variables of $\mathcal{A}_{\mathbb{Q}}(\mathcal{Q})$ can be expressed as Laurent polynomial in $u_{i}$ with positive integer coefficients. By setting $u_{i}=1$ for all $i$ one obtains that all the cluster variables become positive integers. Therefore one gets a frieze with positive integer entries.
However, not all friezes can be obtained this way. For instance in type $\mathbf{D}_{4}$ one has 50 clusters and 51 friezes. The frieze

is not in the image of the above map.
Where do the missing friezes come from?

- Find a combinatorial interpretation of the action of the pentagram map on the 2 -friezes.
- Find an analog of Theorem 2.10 for the action of the pentagram map on the closed $n$-gons.
- Investigate further direction, e.g. higher friezes from higher AR theory, [43]


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# INTEGRABLE SYSTEMS AND FRIEZES: EXERCISE SHEET by Max Glick 

## The pentagram map (to Be discussed Monday)

Problem 1. (Cross ratio identities)
The cross ratio of 4 real numbers is

$$
[a, b, c, d]=\frac{(a-b)(c-d)}{(b-c)(d-a)}
$$

(warning: there are many different conventions for cross ratios and the above definition is unlikely to agree with the one used in lecture). The cross ratio of 4 collinear points in the plane is defined the same way by identifying the line with $\mathbb{R}$ via an affine map.
(a) Show $[a, b, c, d]=[b, a, d, c]=[c, d, a, b]=[d, c, b, a]$.
(b) Suppose $[a, b, c, d]=z$. Calculate each of $[a, b, d, c],[a, c, b, d],[a, c, d, b]$, $[a, d, b, c]$, and $[a, d, c, b]$. Hint: note that $[a, b, c, \cdot]$ is the unique Möbius transformation mapping $a$ to $\infty, b$ to 1 , and $c$ to 0 . Find a similar characterization of [ $a, b, \cdot, c]$ and so on.
(c) (for geometry lovers only) Let $P_{1}, P_{2}, P_{3}, P_{4}$ be points and $L_{1}, L_{2}, L_{3}, L_{4}$ lines in the plane, all in general position. For $1 \leq i<k \leq 4$ let

$$
z_{i, k}=\left[P_{i}, \overleftrightarrow{P_{i} P_{k}} \cap L_{j}, P_{k}, \overleftrightarrow{P_{i} P_{k}} \cap L_{l}\right]
$$

where $\{j, l\}=\{1,2,3,4\} \backslash\{i, k\}$ and $j<l$. Prove that

$$
z_{1,3} z_{2,4}=z_{1,2} z_{2,3} z_{3,4} z_{1,4}
$$

Problem 2. (The $y$-parameters for the pentagram map)
Label the vertices and edges of an $n$-gon in order from 1 to $2 n$. This induces a labeling of its itereates under the pentagram map in a natural way as in


If $j$ is an edge label of a polygon $B$, the corresponding $y$-parameter of $B$ is

$$
y_{j}(B)=-\left[A_{j-2}, B_{j-1}, A_{j+2}, B_{j+1}\right]
$$

where $A=T^{-1}(B)$ (as usual $T$ is the pentagram map). For instance, if $B$ is the inner polygon above then $y_{4}(B)$ equals minus the cross ratio of the 4 indicated points in an appropriate order.

Suppose that $B$ is a polygon, $C=T(B), D=T(C)$, and $j$ is an edge label of $B$. Prove that

$$
\begin{equation*}
y_{j}(B) y_{j}(D)=\frac{\left(1+y_{j-3}(C)\right)\left(1+y_{j+3}(C)\right)}{\left(1+\frac{1}{y_{j-1}(C)}\right)\left(1+\frac{1}{y_{j+1}(C)}\right)} \tag{1}
\end{equation*}
$$

Hint: Geometrically, (1) is an identity of 15 points in the plane (see below). Observe that Problem 1 part (c), involves 16 total points and try to realize the current problem as a special case.


Note: In addition to the edge centered $y$-parameters there are also vertex centered $y$-parameters defined by

$$
\begin{equation*}
y_{j}(B)=\left(y_{j}(C)\right)^{-1} \tag{2}
\end{equation*}
$$

for $j$ a vertex of $B$, where $C=T(B)$. In a certain sense, $y_{1}(B), \ldots, y_{2 n}(B)$ are coordinates on a $2 n$-gon $B$ and (1) and (2) determine how these coordinates transform under the pentagram map.

## Somos sequences (to be discussed Tuesday)

Problem 3. (Quiver for the Somos-4 recurrence)
(a) Find a quiver $Q$ on vertex set $V=\{1,2,3,4\}$ such that

$$
\left(\left(x_{1}, x_{2}, x_{3}, x_{4}\right), Q\right) \xrightarrow{\mu_{1}}\left(\left(\frac{x_{2} x_{4}+x_{3}^{2}}{x_{1}}, x_{2}, x_{3}, x_{4}\right), Q^{\prime}\right)
$$

where $Q^{\prime}$ is obtained from $Q$ by "rotation by 90 degrees". More precisely, $Q \cong Q^{\prime}$ as directed graphs and the isomorphism is induced by the bijection $i \mapsto i+1(\bmod 4)$ on $V$.
(b) Show that the answer to part (a) is unique up to simultaneous reversal of all arrows.
(c) Write down the exchange matrix $B$ associated to $Q$ and calculate its rank.

## Frieze patterns (to be discussed Thursday)

Problem 4. (Determinantal formula for friezes)
(a) Let $\ldots, a_{1}, a_{2}, \ldots$ be the first row below the top row of 1's in a Coxeter-Conway frieze pattern. Prove that the entries of the $k$ th row below the row of 1's are the $k \times k$ determinants

$$
\left|\begin{array}{cccccc}
a_{i} & 1 & 0 & \cdots & 0 & 0 \\
1 & a_{i+1} & 1 & \cdots & 0 & 0 \\
0 & 1 & a_{i+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{i+k-2} & 1 \\
0 & 0 & 0 & \cdots & 1 & a_{i+k-1}
\end{array}\right|
$$

You may use the identity (which is the basis of Dodgon's condensation algorithm)

$$
|M||C|=|N W||S E|-|N E||S W|
$$

where $M$ is an $n \times n$ matrix, $C$ is its central $(n-2) \times(n-2)$ submatrix, and $N W$, $S E, N E, S W$ are its consecutive $(n-1) \times(n-1)$ submatrices in the indicated compass directions.
(b) Let

$$
f_{k}=\left|\begin{array}{cccccc}
a_{0} & 1 & 0 & \cdots & 0 & 0 \\
1 & a_{1} & 1 & \cdots & 0 & 0 \\
0 & 1 & a_{2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{k-2} & 1 \\
0 & 0 & 0 & \cdots & 1 & a_{k-1}
\end{array}\right|
$$

the entries down a diagonal of a frieze starting from $f_{0}=1$ and $f_{1}=a_{0}$. Prove that

$$
f_{k+1}=a_{k} f_{k}-f_{k-1}
$$

Problem 5. (The frieze pattern associated to a triangulation)
Fix a triangulation of an $n$-gon with vertices $0,1, \ldots, n-1$. Let $a_{i}$ for $i \in \mathbb{Z} /(n \mathbb{Z})$ be the number of triangles touching vertex $i$. Define $f_{i}$ for $i=0,1, \ldots, n-1$ recursively by

- $f_{n-1}=0$
- $f_{i}=1$ if $i$ is connected by an edge of the triangulation (or of the polygon) to $n-1$
- whenever the $f$-values at two vertices of a triangle are known, the $f$-value of the third vertex equals their sum.

For example, the following is a triangulation of an octagon, where the vertices are numbered clockwise starting with vertex 0 on the top. The $a_{i}$ are indicated on the left and the $f_{i}$ on the right.

(a) Prove that $f_{0}=1, f_{1}=a_{0}, f_{k+1}=a_{k} f_{k}-f_{k-1}$ for $k=1,2, \ldots, n-3$, and $f_{n-2}=1$.
(b) Prove that $\ldots, a_{0}, a_{1}, \ldots, a_{n-1}, a_{0}, a_{1}, \ldots$ is the top row of a frieze pattern with positive integer entries.


[^0]:    ${ }^{1}$ Note that mutations at vertices not joined by an arrow is commutative so $\mu_{o d d}$ is well defined.

[^1]:    ${ }^{2}$ When representing friezes, one often omits the bordering top and bottom rows of 0 's.

[^2]:    ${ }^{3}$ we haven't defined here the analog of the "tame condition" for the 2-friezes.

[^3]:    ${ }^{4}$ Note that the period of $f_{a d}$ coincides with the Coxeter number associated to the corresponding Dynkin diagram, and the period of $f_{m u}$ is that number plus two.

