International Mathematics Research Notices, 2024, **00**, 1–10
https://doi.org/10.1093/imrn/rnad318
Advance access publication date 00 Month 20xx
Article

Burau Representation of Braid Groups and q-Rationals

Sophie Morier-Genoud[1], Valentin Ovsienko[2] and Alexander P. Veselov[3,*]

[1]Laboratoire de Mathématiques, Université de Reims, U.F.R. Sciences Exactes et Naturelles, Moulin de la Housse - BP 1039, 51687 Reims cedex 2, France
[2]Centre National de la Recherche Scientifique, Laboratoire de Mathématiques, Université de Reims, U.F.R. Sciences Exactes et Naturelles, Moulin de la Housse - BP 1039, 51687 Reims cedex 2, France
[3]Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK
[*]Correspondence to be sent to: e-mail: A.P.Veselov@lboro.ac.uk
Communicated by Prof. Michael Finkelberg

We establish a link between the new theory of q-deformed rational numbers and the classical Burau representation of the braid group $\mathcal{B}_3$. We apply this link to the open problem of classification of faithful complex specializations of this representation. As a result we provide an answer to this problem in terms of the singular set of the q-rationals and prove the faithfulness of the Burau representation specialized at complex $t \in \mathbb{C}^*$ outside the annulus $3 - 2\sqrt{2} \leq |t| \leq 3 + 2\sqrt{2}$.

1 Introduction

The braid groups are the most remarkable groups from topological point of view naturally appearing in the theory of knots, mapping class groups and configuration spaces; see [8]. They were explicitly introduced by Emil Artin in 1925 [3], who denoted their n-strand versions by $\mathcal{B}_n$. The (Artin) braid group $\mathcal{B}_n$ is generated by $n-1$ elements $\sigma_1, \ldots, \sigma_{n-1}$ with braid relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad i = 1, \ldots, n-1,$$

and $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 1$. Since then the braid groups have been extensively studied both by topologists and algebraists.

One of the first important results in this direction was found by Werner Burau in 1936 [10], who introduced what is now known as (reduced) *Burau representation* $\rho_n : \mathcal{B}_n \to \mathrm{GL}(n-1, \mathbb{Z}[t, t^{-1}])$. In the simplest case $n = 3$ the Burau representation $\rho_3 : \mathcal{B}_3 \to \mathrm{GL}(2, \mathbb{Z}[t, t^{-1}])$ is defined by

$$\rho_3 : \quad \sigma_1 \quad \mapsto \quad \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \quad \mapsto \quad \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}, \tag{1}$$

where t is a formal parameter. One can check that the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ is satisfied in this case. Burau used this representation to introduce a knot invariant, which turned out to be essentially the famous Alexander polynomial [1].

It is known after [2, 23] that ρ_3 is *faithful*, meaning that its kernel is trivial. Three different proofs of this fact can be found in [9]. Note that for $n \geq 5$ the Burau representation is known to be non-faithful [5, 22] and that for $n = 4$ the question is still open.

Received: September 19, 2023. **Revised:** November 20, 2023. **Accepted:** December 14, 2023

In this paper we address another open problem (which we call *Burau specialization problem*), which was flagged in a recent paper by Bharathram and Birman (see [9, Open Problem 1 in Section 7]):

"At which complex specializations of t is the Burau representation ρ_3 faithful?"

Here a *specialization* of the Burau representation ρ_3 is the representation

$$\rho_3^t : \mathcal{B}_3 \to \mathrm{GL}(2, \mathbb{C}),$$

which is defined by (1), but t is now a non-zero complex number: $t \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

In the real case with $t \in \mathbb{R}$ some interesting results in this direction were found by Scherich in [33], who used the hyperbolic geometry to prove, in particular, that ρ_3^t is faithful when $t < 0$, $t \neq -1$, and outside the interval $\frac{3-\sqrt{5}}{2} \leq t \leq \frac{3+\sqrt{5}}{2}$. Note that when $t = -1$ the matrices (1) specialize to

$$R := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad L^{-1} := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

generating the whole group $\mathrm{SL}(2, \mathbb{Z})$. Recall also that the quotient of the braid group $\mathcal{B}_3$ by its centre Z (generated by $z := (\sigma_1\sigma_2)^3$) is the classical modular group $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\pm\,\mathrm{Id}$, so that the specialization of the Burau representation ρ_3 at $t = -1$ is not faithful, having Z in its kernel.

The aim of our work is to establish the link of the Burau specialization problem with the theory of q-deformed rational numbers (or q-rationals, for short), which was recently developed in [26]. The q-deformation of a positive rational $\frac{r}{s}$ has the form

$$\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)},$$

where $\mathcal{R}(q)$, $\mathcal{S}(q)$ are certain Laurent polynomials in q with positive integer coefficients (see the details in the next section). The zeros of these polynomials have been studied in [19].

Define the *singular set of q-rationals* $\Sigma \subset \mathbb{C}^*$ as the union of complex poles of all q-rationals and consider the *extended singular set* $\Sigma_* := \Sigma \cup \{1\}$. Both sets Σ and Σ_* consist of certain algebraic integers and are invariant under the involution $q \to q^{-1}$.

The following theorem gives an answer to the Burau specialization problem in terms of q-rationals.

Theorem 1. The Burau representation ρ_3 specialized at $t_0 \in \mathbb{C}^*$ is faithful if and only if $-t_0 \notin \Sigma_*$.

A similar claim in terms of zeros of the Moody polynomials [25] can be found in [9], but our statement allows us to use the theory of q-rationals for which information about the set Σ is available. In particular, from the results of [19] we deduce our second main result.

Theorem 2. The specialized Burau representation $\rho_3^{t_0}$ is faithful for all $t_0 \in \mathbb{C}^*$ outside the annulus $3 - 2\sqrt{2} \leq |t_0| \leq 3 + 2\sqrt{2}$.

We believe that we can reduce the annulus in Theorem 2 as follows.

Conjecture 3. The specialized Burau representation $\rho_3^{t_0}$ is faithful for all $t_0 \in \mathbb{C}^*$ outside the annulus $\frac{3-\sqrt{5}}{2} \leq |t| \leq \frac{3+\sqrt{5}}{2}$.

Modulo Theorem 1 this conjecture is a weaker version of the conjecture from [19], which claims, in particular, that Σ_* lies within the annulus $\frac{3-\sqrt{5}}{2} \leq |q| \leq \frac{3+\sqrt{5}}{2}$.

Note that the bounds $\frac{3-\sqrt{5}}{2}$ and $\frac{3+\sqrt{5}}{2}$ were also found in [33] in the real case, which confirms this conjecture for $t_0 \in \mathbb{R}^*$.

Our approach is based on the following important observation. Consider the natural projective version of the Burau representation

$$\hat{\rho}_3 : \mathcal{B}_3 \to \mathrm{PGL}(2, \mathbb{Z}[t, t^{-1}]) := \mathrm{GL}(2, \mathbb{Z}[t, t^{-1}])/\{\pm t^n \mathrm{Id},\ n \in \mathbb{Z}\}. \tag{2}$$

Note that the centre $Z \subset \mathcal{B}_3$ is in the kernel of $\hat{\rho}_3$ since $\rho_3((\sigma_1\sigma_2)^3) = t^3\mathrm{Id}$. This means that image $\hat{\rho}_3(\mathcal{B}_3)$ is isomorphic to the modular group $\mathrm{PSL}(2, \mathbb{Z})$.

Our key observation is that this image coincides with the q-deformation $\mathrm{PSL}(2,\mathbb{Z})_q$ of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ introduced in [26] (see also [18]) if we identify t with $-q$. Indeed, $\mathrm{PSL}(2,\mathbb{Z})_q$ is generated by

$$\begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 1 & q^{-1} \end{pmatrix},$$

where q is a formal parameter, which are projectively equivalent to $\rho_3(\sigma_1)$ and $\rho_3(\sigma_2)^{-1}$ respectively when $q = -t$.

We should note that the theory of q-rationals was initially motivated by connections to cluster algebras and rapidly led to further developments in various directions: combinatorics of posets, knot invariants, Markov numbers and Diophantine analysis, enumerative geometry, triangulated categories and homological algebra, quantum calculus (see more on this in the review [29]). It is interesting that although the knot theory was already discussed in this connection, as far as we know, no relation with the Burau representation ρ_3 was pointed out so far.

2 q-Rationals and q-Deformed Modular Group

In this section we give the precise and adapted for computations definition of the q-rationals, and recall their main properties, which are necessary for the proof of the main result. Among several equivalent definitions of q-rationals we will use the one based on $\mathrm{PSL}(2,\mathbb{Z})$-invariance (or on $\mathcal{B}_3$-invariance).

2.1 The main definition

Consider the set $\mathbb{Q} \cup \{\infty\}$ rational numbers, extended by one additional element ∞, which will always be represented by the quotient $\frac{1}{0}$. The group $\mathrm{SL}(2,\mathbb{Z})$ of matrices with integer coefficients

$$M = \begin{pmatrix} r & v \\ s & u \end{pmatrix}, \qquad r, v, s, u \in \mathbb{Z}, \quad ru - vs = 1,$$

acts on $\mathbb{Q} \cup \{\infty\}$ by linear-fractional transformations:

$$M \cdot x = \frac{rx + v}{sx + u}. \tag{3}$$

This action is homogeneous and effective for the modular group $\mathrm{PSL}(2,\mathbb{Z})$.

Following [26], consider the following matrices of $\mathrm{GL}(2,\mathbb{Z}[q,q^{-1}])$:

$$R_q := \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \qquad L_q := \begin{pmatrix} 1 & 0 \\ 1 & q^{-1} \end{pmatrix}. \tag{4}$$

They generate a subgroup $\mathrm{PSL}(2,\mathbb{Z})_q$ in the group

$$\mathrm{PGL}(2,\mathbb{Z}[q,q^{-1}]) = \mathrm{GL}(2,\mathbb{Z}[q,q^{-1}])/\{\pm q^n \mathrm{Id}\},$$

which is isomorphic to $\mathrm{PSL}(2,\mathbb{Z})$ and thus can be considered as *q-deformed modular group* [18, 26]. Evaluating at $q = 1$ one recovers $R = R_q|_{q=1}$ and $L = L_q|_{q=1}$ the standard generators of the modular group $\mathrm{SL}(2,\mathbb{Z})$. This allows one to consider q-analogs of matrices M_q for every $M \in \mathrm{SL}(2,\mathbb{Z})$.

We have a natural action of $\mathrm{PSL}(2,\mathbb{Z})_q$ on the space $\mathbb{Z}(q)$ of rational functions in q with integer coefficients, which can be considered as q-deformed version of the standard linear-fractional action of the modular group. More precisely, for every $M \in \mathrm{SL}(2,\mathbb{Z})$, written as a monomial in R and L, let M_q be a matrix in $\mathrm{GL}(2,\mathbb{Z}[q,q^{-1}])$ given by the same monomial in R_q and L_q. If $f(q) \in \mathbb{Z}(q)$, then the linear-fractional action

$$f(q) \mapsto M_q \cdot f(q) \tag{5}$$

is a well-defined action of $\mathrm{PSL}(2,\mathbb{Z})$. In other words, M_q does depend on the choice of the monomial in R and L presenting M, but the result of (5) does not.

Definition 1. The set of q-rationals is the set of rational functions in the orbit of any point from the set $\{0, 1, \infty\}$ (all of them remain independent of q after deformation) for the linear-fractional action (5).

Example 2. Every rational is an image of 0 under the action (3). For instance,

$$1 = R\cdot 0, \quad 2 = R^2\cdot 0, \quad \frac{1}{2} = LR\cdot 0, \quad \frac{2}{3} = LR^2\cdot 0\,.$$

The element of $\mathrm{SL}(2,\mathbb{Z})$ corresponding to a rational is determined by its continued fraction expansion. Therefore, one obtains

$$R_q\cdot 0 = 1, \quad R_qR_q\cdot 0 = 1+q, \quad L_qR_q\cdot 0 = \frac{q}{1+q}, \quad L_qR_qR_q\cdot 0 = \frac{q+q^2}{1+q+q^2}, \quad \ldots$$

as the q-analogs of $1, 2, \frac{1}{2}, \frac{2}{3}$, respectively.

Remark 3. The notion of q-rationals extends that of q-integers:

$$\begin{array}{rcl} [n]_q & := & 1+q+q^2+\ldots+q^{n-1} \\ [-n]_q & := & -q^{-1}-q^{-2}\ldots-q^{-n}, \end{array}$$

that goes back to the works of Euler and Gauss. These q-integers and the corresponding q-binomial coefficients are essential in quantum algebra and mathematical physics. They are also the key ingredients for the theory of q-analogs in combinatorics. Most classical sequences of integers have interesting q-analogs often arising as generating functions.

2.2 Matrices of continued fractions

We use the following notation for the entries and the decomposition in terms of generators of a matrix M in $\mathrm{SL}(2,\mathbb{Z})$

$$M \quad = \quad \begin{pmatrix} r & v \\ s & u \end{pmatrix} \quad = \quad R^{a_1}L^{a_2}\cdots R^{a_{2m-1}}L^{a_{2m}} \quad =: \quad M^+(a_1,\ldots,a_{2m}), \tag{6}$$

and the following for the corresponding q-deformed matrices in $\mathrm{GL}(2,\mathbb{Z}[q,q^{-1}])$:

$$M_q \quad = \quad \begin{pmatrix} \mathcal{R} & \mathcal{V} \\ \mathcal{S} & \mathcal{U} \end{pmatrix} \quad = \quad R_q^{a_1}L_q^{a_2}\cdots R_q^{a_{2m-1}}L_q^{a_{2m}} \quad =: \quad M_q^+(a_1,\ldots,a_{2m}), \tag{7}$$

where a_i and r, s, t, u are integers, and $\mathcal{R}, \mathcal{S}, \mathcal{V}, \mathcal{U}$ are Laurent polynomials in q. This notation is very convenient for explicit computations, taking into account that the coefficients a_i are those of the continued fraction expansion $\frac{r}{s} = [a_1,\ldots,a_{2m}]$.

The following definition is equivalent to Definition 1.

Definition 4 ([26]). If $\frac{r}{s}$ is a rational appearing in the first column of a matrix M we define its q-analog as the rational function given by

$$\left[\frac{r}{s}\right]_q := \frac{\mathcal{R}(q)}{\mathcal{S}(q)}, \tag{8}$$

where $\mathcal{R}(q)$ and $\mathcal{S}(q)$ are the entries in the first column of the corresponding matrix M_q.

Example 5. From

$$M^+(1,1) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_q^+(1,1) = \begin{pmatrix} 1+q & q^{-1} \\ 1 & q^{-1} \end{pmatrix},$$

one gets $[2]_q = 1+q$, which is the standard quantum integer.
From

$$M^+(0,2) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad M_q^+(0,2) = \begin{pmatrix} 1 & 0 \\ 1+q^{-1} & q^{-2} \end{pmatrix},$$

one gets $\left[\frac{1}{2}\right]_q = \frac{q}{1+q}$.
From

$$M^+(0,1,1,1) = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \quad M_q^+(0,1,1,1) = \begin{pmatrix} 1+q & q^{-1} \\ q+1+q^{-1} & q^{-1}+q^{-2} \end{pmatrix},$$

one gets $\left[\frac{2}{3}\right]_q = \frac{q+q^2}{1+q+q^2}$.

2.3 Some properties of the polynomials $\mathcal{R}$ and $\mathcal{S}$

Let as before $\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$. We collect some of the known properties of the polynomials $\mathcal{R}(q)$ and $\mathcal{S}(q)$ that will be useful for the proof. We were not able to find these properties in the literature about the Burau representation.

Suppose first that $\frac{r}{s} \geq 1$.

Proposition 6 (Positivity [26, Prop. 1.3]). The polynomials $\mathcal{R}$ and $\mathcal{S}$ have positive integer coefficients.

Note that this positivity statement can be strengthened, cf. Theorem 2 of [26].

Proposition 7 (Reflection and mirror [18, Prop 2.8]). One has

$$\left[\frac{s}{r}\right]_q = \frac{\mathcal{S}(q^{-1})}{\mathcal{R}(q^{-1})} \qquad \text{and} \qquad \left[-\frac{r}{s}\right]_q = -\frac{\mathcal{R}(q^{-1})}{q\mathcal{S}(q^{-1})}.$$

Proposition 8 (Roots of polynomials [19, Section 5.1]). The polynomial $\mathcal{R}$ and $\mathcal{S}$ have no roots inside the punctured disc with radius $3-2\sqrt{2}$.

Proposition 8 can be extended for an arbitrary rational $\frac{r}{s} < 1$ using $\mathrm{PSL}(2,\mathbb{Z})$-action. Proposition 7 implies that the polynomials $\mathcal{R}$ and $\mathcal{S}$ have no roots for $|q| \geq \frac{1}{3-2\sqrt{2}} = 3+2\sqrt{2}$. Therefore, one gets the following corollary.

Proposition 9. For every rational $\frac{r}{s}$ the roots of the polynomials $\mathcal{R}$ and $\mathcal{S}$ belong to the open annulus $\{3-2\sqrt{2} < |t| < 3+2\sqrt{2}\} \subset \mathbb{C}^*$.

3 Proof of the Main Theorems

The following important observation links the theory of q-rationals with Burau representation.

Proposition 10. The q-deformed action of the modular group (5) coincides with the projective version of the Burau representation (2) with $q = -t$.

Proof. Indeed, for $t = -q$ the matrices R_q and L_q given by (4) coincide with $\rho_3(\sigma_1)$ and $\rho_3(\sigma_2)^{-1}$. ■

This means that from the point of view of Burau representation, q-rationals are the quotients of the elements in the columns of the matrices $\rho_3(\beta)$, for every $\beta \in \mathcal{B}_3$, but the sign of the formal parameter is reversed. More precisely, if the Burau representation written in the matrix form

$$\rho_3(\beta) = \begin{pmatrix} \mathcal{R}(t) & \mathcal{V}(t) \\ \mathcal{S}(t) & \mathcal{U}(t) \end{pmatrix}, \tag{9}$$

then $\frac{\mathcal{R}(q)}{\mathcal{S}(q)}$ and $\frac{\mathcal{V}(q)}{\mathcal{U}(q)}$ with $q = -t$ are q-rationals.

Example 11. Take $\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$, this braid corresponds to the continued fraction $[1, 1, 1, 1] = \frac{5}{3}$. As explained in the beginning of Section 2.2, the rational $\frac{5}{3}$ then appears as the quotient of the elements in the first column of the matrix $RLRL$. The Burau representation gives the matrix

$$\rho_3(\beta) = t^{-2}\begin{pmatrix} -t + t^2 - 2t^3 + t^4 & 1 - t + t^2 \\ -t + t^2 - t^3 & 1 - t \end{pmatrix} = q^{-2}\begin{pmatrix} q + q^2 + 2q^3 + q^4 & 1 + q + q^2 \\ q + q^2 + q^3 & 1 + q \end{pmatrix}. \tag{10}$$

so that the rational function $\frac{1+q+2q^2+q^3}{1+q+q^2}$ is the q-deformed $\frac{5}{3}$. Similarly, $\frac{1+q+q^2}{1+q} =: \left[\frac{3}{2}\right]_q$ in the second column.

Now we can use the results of [19] about the roots of the polynomials $\mathcal{R}, \mathcal{S}, \mathcal{V}$ and $\mathcal{U}$ to prove our main results.

We start with the following lemma.

Lemma 12. Assume that in (9) matrix element $S(t) \equiv 0$, then β belongs to the subgroup $G_0 \subset \mathcal{B}_3$ generated by σ_1 and $(\sigma_1\sigma_2)^3$.

Proof. Specialization at $t = -1$ gives a matrix of the triangular form

$$\rho_3^{-1}(\beta) = \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}),$$

which is the same as $\rho_3^{-1}(\sigma_1^k)$ for some $k \in \mathbb{Z}$. Since the kernel of this specialization is the centre Z of $\mathcal{B}_3$ generated by $(\sigma_1\sigma_2)^3$, we have the claim. ∎

Note the converse of the lemma is also obviously true.

Consider now the set $\Sigma \subset \mathbb{C}^*$ defined as the union of the poles of all q-rationals $\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$, $\frac{r}{s} \in \mathbb{Q}$. Note that since

$$\left[\frac{1}{n}\right]_q = \frac{q^{n-1}(1-q)}{1-q^n}$$

Σ contains all roots of unity except 1 itself. Recall that $\Sigma_* = \Sigma \cup \{1\}$ is the extended set with added special point $q = 1$.

We are ready to prove Theorem 1, that is, to prove that the Burau representation ρ_3 specialized at $t_0 \in \mathbb{C}^*$ is faithful if and only if $-t_0 \notin \Sigma_*$.

Proof. Let us first prove that if $-t_0 \notin \Sigma_*$ then $\rho_3^{t_0}$ is faithful. Given a braid $\beta \in \mathcal{B}_3$ we need to show that $\rho_3^{t_0}(\beta) = \mathrm{Id}$ implies that β is a trivial braid.

Assume that $\rho_3^{t_0}(\beta) = \mathrm{Id}$. Proposition 10 implies that the polynomials $\mathcal{S}$ and $\mathcal{V}$ in the matrix (9) vanish identically: $\mathcal{S}(t) = \mathcal{V}(t) \equiv 0$, since t_0 is not a root of these polynomials. Therefore,

$$\rho_3(\beta) = \begin{pmatrix} \mathcal{R}(t) & 0 \\ 0 & \mathcal{U}(t) \end{pmatrix}.$$

We know that $\rho_3(\beta)$ evaluated at $t_0 = -1$ belongs to $\mathrm{SL}(2,\mathbb{Z})$. Proposition 6 then implies that the polynomials $\mathcal{R}$ and $\mathcal{U}$ are monomials, that is, $\mathcal{R}(t) = \pm t^{\ell}$ and $\mathcal{U}(t) = \pm t^{k}$, for some integers ℓ and k. Since, by assumption t_0 is not a root of 1 or -1, $\rho_3^{t_0}(\beta) = \mathrm{Id}$ implies that the polynomials $\mathcal{R}$ and $\mathcal{U}$ are constant equal to 1. Finally one has $\rho_3(\beta) = \mathrm{Id}$ and therefore β is a trivial braid because the Burau representation ρ_3 is faithful.

To prove the converse statement we use some ideas from [33]. We need to prove that if $-t_0 \in \Sigma_*$ then $\rho_3^{t_0}$ is unfaithful. First of all, we know that this is true if $t_0 = -1$. Assume now that $-t_0 \in \Sigma$ and consider a braid $\beta_0 \in \mathcal{B}_3$ such that the corresponding element $\mathcal{S}(t)$ in (9) is not identically zero, but $\mathcal{S}(t_0) = 0$. Assume that the specialization $\rho_3^{t_0}$ is faithful and consider the subgroup G of $\mathcal{B}_3$ generated by σ_1 and β_0. Since both $\rho_3^{t_0}(\sigma_1)$ and $\rho_3^{t_0}(\beta_0)$ are triangular, G must be solvable, but not abelian (since β_0 does not commute with σ_1 because $\mathcal{S}(t) \not\equiv 0$).

Let us prove that this is impossible in our case. (We are very grateful to A.Yu. Olshanskiy for the help with this proof.) Introducing $x = \sigma_1\sigma_2$ and $y = \sigma_2\sigma_1\sigma_2$ we can rewrite the generating relation as $y^2 = x^3$. The centre Z is generated by $x^3 = y^2$, so the quotient $\mathcal{B}_3/Z = \mathbb{Z}_2 * \mathbb{Z}_3$ is the free product of two cyclic groups of order 2 and 3 respectively (which is also isomorphic to the modular group $\mathrm{PSL}(2,\mathbb{Z})$). Consider the image $\phi(G) \subset \mathbb{Z}_2 * \mathbb{Z}_3$ under the homomorphism $\phi : \mathcal{B}_3 \to \mathcal{B}_3/Z$. From Kurosh subgroup theorem [16] it follows that any solvable subgroup of $\mathbb{Z}_2 * \mathbb{Z}_3$ is either cyclic, or infinite dihedral group $D_\infty = \mathbb{Z}_2 \ltimes \mathbb{Z}$, which is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$ (generated by the images of some conjugates of y). In the first case G is abelian. We claim that the second case is not possible as well. Indeed, if $G = D_\infty$, then it must belong to the kernel of the homomorphism $\psi : \mathcal{B}_3 \to \mathbb{Z}_3$ sending $y \to 0$ and $x \to 1$ (in the additive notation). Since $\psi(\sigma_1) = \psi(x^2y^{-1}) = 2 \neq 0$, we have a contradiction.

Theorem 1 is proved. ■

Now Theorem 2 is an immediate corollary of Theorem 1 and Proposition 9. Note also that Proposition 6 implies the result of [33] about faithfulness of ρ_3^t for negative $t \neq -1$.

4 Discussion

Perhaps the most interesting question now is to understand the number-theoretic properties of the algebraic integers from the set Σ. The following results from the theory of q-rationals can give more information in this direction.

• The sequences of coefficients in the polynomials written in the parameter q are unimodal. This fact, which was conjectured in [26] and studied in [24], was eventually proved in [28].

This implies, in particular, that if an algebraic integer $\alpha \in \mathbb{C}$ has a conjugate, which is real and positive, then α cannot be part of Σ (and hence, corresponding specialization of the Burau representation at $t_0 = -\alpha$ is faithful).

• The trace of the matrix (9) is a palindromic polynomial (in variable q) [18]. We wonder if this is related to the invariant Hermitian form of [34].

• The "stabilization phenomenon" of [27] is one of the main properties of q-rationals. Suppose that a sequence of rationals, $\frac{r_m}{s_m}$, converges to an irrational number x. Then the Taylor series of the rational functions $\frac{\mathcal{R}_m(q)}{\mathcal{S}_m(q)}$ stabilizes, as m grows. This allows to define a q-deformation $[x]_q$ as Taylor series with integer coefficients. The study of the radii of convergence of these series is closely related to the study of the singular set Σ. In particular if $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, then the radius of convergence of the corresponding series $[\varphi]_q$ is $R(\varphi) = \frac{3-\sqrt{5}}{2}$ (see [19]).

4.1 The conjectural tight limits of faithfulness

The annulus in Theorem 2 can be reduced to $\frac{3-\sqrt{5}}{2} \le |t| \le \frac{3+\sqrt{5}}{2}$ modulo the following conjecture.

Conjecture 2 ([19]). For every real $x > 0$ the radius of convergence $R(x)$ of the series $[x]_q$ satisfies the inequality

$$R(x) \ge R(\varphi) = \frac{3-\sqrt{5}}{2}$$

and the equality holding only for x, which are $\mathrm{PSL}(2,\mathbb{Z})$-equivalent to φ.

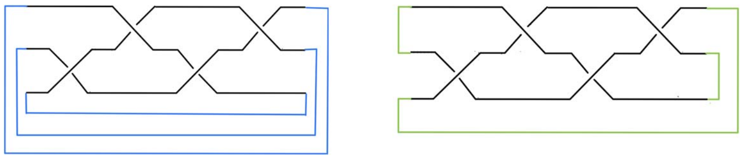

Fig. 1.. Two different closures of $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$

Since for a q-rational $\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$ the radius of convergence equals the minimal modulus of the roots of the denominator $\mathcal{S}(q)$, modulo this conjecture we can claim that Σ_* lies within the annulus $\frac{3-\sqrt{5}}{2} \le |q| \le \frac{3+\sqrt{5}}{2}$ and therefore the specialized Burau representation ρ_3^t is faithful for all t outside the annulus $\frac{3-\sqrt{5}}{2} \le |t| \le \frac{3+\sqrt{5}}{2}$.

In some special cases Conjecture 2 was proved in [19, 31]. Computer experiments show that the bounds $\frac{3\pm\sqrt{5}}{2}$ for the annulus in the conjectures are optimal. In particular, the polynomials in the entries of the matrices $M_q^+(1, 1, \dots, 1) = (R_q L_q)^m$ have roots closer and closer to the circle $|t| = \frac{3-\sqrt{5}}{2}$, as m grows.

4.2 Braids, rational knots, and their invariants

There exist two different ways to associate a knot (or a link) to $\beta \in \mathcal{B}_3$: one is by the standard closure shown on the left of Figure 1 and the second one by the closure (used by Conway in [11]), which is shown on the right of this Figure. In the concrete shown case of $\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ these two closures lead to the same knot (which is the famous *figure-eight knot*), but in general this is not the case.

To obtain the *rational (or two-bridge) knots* (see, e.g., Lickorish [21]) from the braids in $\mathcal{B}_3$ one should use the second closure (to get them using the standard closure one may need four-strand braid group $\mathcal{B}_4$).

As we have already mentioned in the Introduction, the Alexander polynomial of the link $L = L(\beta)$ related to a braid $\beta \in \mathcal{B}_n$ via the standard closure can be given (up to a unit in $\mathbb{Z}[t, t^{-1}]$) by the Burau formula [10]

$$\Delta_L(t) = \frac{1-t}{1-t^n} \det(I - \rho_n(\beta)).$$

In particular, using formula (10) we get the Alexander polynomial for the figure-eight knot:

$$\Delta_L(t) = -t^{-1} + 3 - t.$$

No general formula in terms of Burau representation is known for the Jones polynomials. However, as it was noticed in [26], for the rational knots the (normalized) Jones polynomial of the knot corresponding to a fraction $\frac{r}{s}$ can be expressed using the polynomials $\mathcal{R}(q)$ and $\mathcal{S}(q)$ from the q-version of $\frac{r}{s}$ in (8) via

$$J_{\frac{r}{s}}(q) = q\mathcal{R}(q) + (1-q)\mathcal{S}(q) \tag{11}$$

(see Appendix A in [26]). In particular, in our case of figure-eight knot we have $r = 5, s = 3$ (or, equivalently, in Conway's convention $r = 5$, $s = 2$) and thus from (10) the Jones polynomial is

$$J_{\frac{5}{3}}(q) = q^{-2}[q(1+q+2q^2+q^3) + (1-q)(1+q+q^2)] = q^{-2} + q^{-1} + 1 + q + q^2$$

in agreement with [21].

The proof of formula (11) is based on the results of the paper [20] by Lee and Schiffler, using the combinatorics of snake graphs and cluster algebras. Alternatively, in terms of the so-called left version of the q-deformed rationals

$$\left[\frac{r}{s}\right]_q^\flat = \frac{\mathcal{R}^\flat(q)}{\mathcal{S}^\flat(q)},$$

which was observed in [27] (cf. Remark 2.3) and introduced and studied by Bapat et al [4], one can express the normalized Jones polynomial as $J_{\frac{r}{s}}(q) = q^m \mathcal{R}^\flat(q^{-1})$ with some $m \in \mathbb{Z}$ (see Appendix A2 in [4] and [32]).

It would be interesting to see if all this can be explained using Burau representation. In this relation we would like to note that the role of the left q-rationals became more clear after the recent preprint [35], where an infinitesimal analogue of the Burau representation is investigated.

4.3 Other representations and specializations

The Burau representation ρ_n fails to be faithful for $n \geq 5$, but there is another homological representation of the braid group $\mathcal{B}_n$, first introduced in [17] and now known as Lawrence–Krammer–Bigelow representation, which is proved to be faithful for all n [6, 15]. The work of Lawrence [17] was initially motivated by the desire to better understand the significance of the Jones polynomial for links [12]. Burau representation corresponds to the first non-trivial case $m = 1$ in her construction. The general case is related also to the monodromy representations for the Knizhnik–Zamolodchikov equations

$$\kappa \frac{\partial \psi}{\partial z_i} - \sum_{k \neq i} \frac{\Omega_{ik}}{z_i - z_k} \psi(z_1, \ldots, z_n) = 0, \quad i = 1, \ldots, n,$$

which were studied by Tsuchiya and Kanie [36] and Kohno [14].

This makes an important link of the braid groups and knots with the theory of Yang–Baxter equations and quantum integrable systems (see more on this in [13, 37]). There is a well-known representation [12] of the braid group in the Temperley–Lieb algebra, which appeared in a similar relation in statistical mechanics. Bigelow [7] proved the equivalence of the faithfulness problem for Burau, Jones and Temperley–Lieb representations in the most interesting case $n = 4$. It is known that the Temperley–Lieb representation of the braid group $\mathcal{B}_3$ is faithful [30], but the same question about its specializations seems to be open. It would be interesting to apply our approach to this problem as well.

Finally, we would like to mention that the specializations of Burau representations of $\mathcal{B}_n$ at the roots of unity naturally appear in algebraic geometry as the homological monodromy in the moduli space of algebraic curves

$$y^d = x^n + a_1 x^{n-1} + \cdots + a_0.$$

This observation goes back to the work of Arnold [2], who considered the hyperelliptic case $d = 2$, and was explicitly stated by Magnus and Peluso [23] in the general case. A more recent important development in this direction is due to Venkataramana [38], who proved the arithmeticity of the image of the Burau representation ρ_n specialized at the d-th roots of unity when $n \geq 2d + 1$.

References

1. Alexander, J. W. "Topological invariants of knots and links." *Trans. Amer. Math. Soc.* **30** (1928): 275–306. https://doi.org/10.1090/S0002-9947-1928-1501429-1.
2. Arnold, V. I. "Remark on the branching of hyperelliptic integrals as functions of the parameters." *Funktsional. Anal. Prilozhen.* **2** (1968): 1–3.
3. Artin, E. "Theorie der Zöpfe." *Abh. Math. Sem. Univ. Hamburg* **4** (1925): 47–72. https://doi.org/10.1007/BF02950718.
4. Bapat, A., Becker, L., and Licata, A. q-deformed rational numbers and the 2-Calabi–Yau category of type A_2.*Forum Math. Sigma* **11** (2023), Paper No. e47.
5. Bigelow, S. "The Burau representation is not faithful for n = 5." *Geom. Topol.* **3**, no. 1 (1999): 397–404. https://doi.org/10.2140/gt.1999.3.397.
6. Bigelow, S. "Braid groups are linear." *J. Amer. Math. Soc.* **14** (2001): 471–86.
7. Bigelow, S. "Does the Jones polynomial detect the unknot?" *J. Knot Theory Ramifications* **11** (2002): 493–505. https://doi.org/10.1142/S0218216502001779.
8. Birman, J. *Braids, Links, and Mapping Class Groups* Ann. Math. Stud, Vol. **82**. Princeton University Press, 1974.
9. Bharathram, V., and Birman, J. On the Burau representation of B_3. arXiv:2208.12378.

10. Burau, W. “Über Zopfgruppen und gleichsinnig verdrillte Verkettungen.” *Abh. Math. Sem. Univ. Hamburg* **11** (1936): 179–86. https://doi.org/10.1007/BF02940722.
11. Conway, J. H. “An enumeration of knots and links, and some of their algebraic properties.” *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*. 329–58. Oxford-New York-Toronto, Ont.: Pergamon, 1970.
12. Jones, V. F. R. “Hecke algebra representations of braid groups and link polynomials.” *Ann. of Math. (2)* **126**, no. 2 (1987): 335–88.
13. Kassel, Ch., Turaev, V.G. *Braid Groups* . Springer, 2008, https://doi.org/10.1007/978-0-387-68548-9.
14. Kohno, T. “Hecke algebra representations of braid groups and classical Yang-Baxter equations.” *Adv. Stud. Pure Math.* **16** (1988): 255–69.
15. Krammer, D. “Braid groups are linear.” *Ann. of Math. (2)* **155** (2002): 131–362.
16. Kurosh, A. “Die Untergruppen der freien Produkte von beliebigen Gruppen.” *Math. Annalen* **109** (1934): 647–60. https://doi.org/10.1007/BF01449159.
17. Lawrence, R. J. “Homological representations of the Hecke algebra.” *Comm. Math. Phys.* **135**, no. 1 (1990): 141–91. https://doi.org/10.1007/BF02097660.
18. Leclere, L., and S. Morier-Genoud. “q-deformations in the modular group and of the real quadratic irrational numbers.” *Adv. Appl. Math.* **130** (2021). https://doi.org/10.1016/j.aam.2021.102223.
19. Leclere, L., S. Morier-Genoud, V. Ovsienko, and A. Veselov. “On radius of convergence of q-deformed real numbers.” *Mosc. Math. J.* to appear, arXiv:2102.00891.
20. Lee, K., and R. Schiffler. “Cluster algebras and Jones polynomials.” *Selecta Math. (N.S.)* **25** (2019). https://doi.org/10.1007/s00029-019-0503-x.
21. Lickorish, W.B.R. *An Introduction to Knot Theory* . Springer New York, Inc., 1997, https://doi.org/10.1007/978-1-4612-0691-0.
22. Long, D. D., and M. Paton. “The Burau representation is not faithful for n $\geq$ 6.” *Topology* **32**, no. 2 (1992): 439–47.
23. Magnus, W., and A. Peluso. “On a theorem of V.” *I. Arnol’d. Comm. Pure Appl. Math.* **22** (1969): 683–92. https://doi.org/10.1002/cpa.3160220508.
24. McConville, T., B. E. Sagan, and C. Smyth. “On a rank-unimodality conjecture of Morier-Genoud and Ovsienko.” *Discrete Math.* **344** (2021): 8.
25. Moody, J. A. “The Burau representation of the braid group *B_n* is unfaithful for large *n*.” *Bulletin of the AMS* **25**, no. 2 (1991): 379–84.
26. Morier-Genoud, S., and V. Ovsienko. “q-deformed rationals and q -continued fractions.” *Forum Math. Sigma* **8** (2020) e13, 55 pp.
27. Morier-Genoud, S., and V. Ovsienko. “On q-deformed real numbers.” *Experiment. Math.* **31** (2022): 652–60. https://doi.org/10.1080/10586458.2019.1671922.
28. Oguz, E. K., and M. Ravichandran. “Rank polynomials of fence posets are unimodal.” *Discrete Math.* **346**, no. 2 (2023) Paper No. 113218.
29. Ovsienko, V. *Modular invariant q-deformed numbers: first steps* .https://amathr.org/modular-invariant-q-deformed-numbers-first-steps/.
30. Piwocki, A.H. A new proof of the faithfulness of the Temperley-Lieb representation ofB_3.*Demonstratio Mathematica XLIV* (2011).
31. Ren, X. “On radiuses of convergence of q-metallic numbers and related q-rational numbers.” *Res. Number Theory* **8** (2022). https://doi.org/10.1007/s40993-022-00336-7.
32. Ren, X. “On q-deformed Farey sum and a homological interpretation of q-deformed real quadratic irrational numbers.” arXiv:2210.06056.
33. Scherich, N. Classification of the real discrete specializations of the Burau representation ofB_3.*Math. Proc. Camb. Phil. Soc.* **168**, 2020.
34. Squier, C. “The Burau representation is unitary.” *Proc. Amer. Math. Soc.* **90** (1984): 199–202. https://doi.org/10.1090/S0002-9939-1984-0727232-8.
35. Thomas, A. “Infinitesimal modular group: q-deformed sl_2*and Witt algebra.*” arXiv:2308.06158.
36. Tsuchiya, A., and Y. Kanie. “*Vertex operators in conformal field theory on*$\mathbb{P}^1$*and monodromy representations of braid groups*.” *Adv. Stud. Pure Math.* **16** (1988): 297–372.
37. Turaev, V. G. “The Yang-Baxter equation and invariants of links.” *Invent. Math.* **92** (1988): 527–53. https://doi.org/10.1007/BF01393746.
38. Venkataramana, T. N. “Image of the Burau representation at d-th roots of unity.” *Ann. of Math. (2)* **179** (2014): 1041–83. https://doi.org/10.4007/annals.2014.179.3.4.