

On q -deformed Markov numbers. Cohn matrices and perfect matchings with weighted edges

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Abstract

We consider a natural q -deformation of the classical Markov numbers. This q -deformation is closely related to q -deformed rational numbers recently introduced by two of us. Both notions, those of q -rationals and q -Markov numbers, are based on invariance with respect to the action of the modular group $\mathrm{PSL}(2, \mathbb{Z})$. We prove that every Markov number has a unique q -deformation, which is a monic unimodal palindromic Laurent polynomial with positive integer coefficients. The q -Markov numbers can be calculated in terms of the traces of q -deformed Cohn matrices, and we show that q -Markov numbers are independent of the choice of such matrices. We construct a combinatorial model counting perfect matchings of snake graphs with weighted edges.

1 Introduction

A classical *Markov triple* is a triple (a, b, c) of positive integers satisfying the Diophantine equation

$$a^2 + b^2 + c^2 = 3abc, \quad (1)$$

called the Markov equation. The *Markov numbers* are the positive integers appearing in a Markov triple. The sequence of Markov numbers (sequence A002559 in [32]) starts as follows

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, \dots$$

Discovered in [24], Markov numbers play a crucial role in many branches of mathematics, from number theory to topology, combinatorics, and mathematical physics. The celebrated Markov, or Frobenius [15] *uniqueness conjecture* affirms that every Markov number appears exactly once as the maximal number of a Markov triple. We refer to [1, 34] for excellent overviews of the status of this conjecture, and, more generally, for the Markov theory.

The goal of this paper is to study the q -analogue of the Markov equation

$$a_q^2 + b_q^2 + c_q^2 = q^{-1} [3]_q a_q b_q c_q + (q - 1)(q^{-1} - 1), \quad (2)$$

where $[3]_q = q^2 + q + 1$ and q is a formal variable. We will be considering Laurent polynomials in q with integer coefficients that appear in a triple $(a_q, b_q, c_q) \in \mathbb{Z}[q^{\pm 1}]^3$, satisfying (2). We will furthermore assume that substituting $q = 1$ we obtain a Markov triple of positive integers:

$$(a_q, b_q, c_q) \Big|_{q=1} = (a, b, c).$$

In this sense, we understand the triple of polynomials (a_q, b_q, c_q) as a q -deformation of the Markov triple (a, b, c) . Note that we do not assume a-priori any property of coefficients of Laurent polynomials, except that they are integers.

Our study of the equation (2) is motivated, in part, by a natural appearance of this equation in the context of q -deformed rationals. The solutions to (2) are polynomials with many beautiful properties such as positivity, palindromicity, and unimodality. Bearing in mind the idea that a q -deformation retains all of the information about the initial object, but often with more detail, it is always interesting to consider natural q -analogues. For instance, the uniqueness conjecture is clearly true in the q -deformed case. This

paper is a step in the study of the q -analogue of the Markov equation. We expect further development of the subject and, in particular, more connections to other topics, such as the theory of dimer coverings.

The first result of this paper is the following existence and uniqueness statement that will be proved in Section 3.

Theorem 1. *For every Markov triple $(a, b, c) \in \mathbb{Z}_{>0}^3$ there exists a unique triple of Laurent polynomials with integer coefficients $(a_q, b_q, c_q) \in \mathbb{Z}[q^{\pm 1}]^3$ which are solutions to (2) such that $(a_q, b_q, c_q)|_{q=1} = (a, b, c)$.*

The proof of Theorem 1 is based on the following recurrent procedure that provides one way to calculate the q -Markov numbers explicitly. This recurrence is very similar to the classical case.

Proposition 2. *Every q -Markov triple (a_q, b_q, c_q) can be obtained from the triple $(1, 1, 1)$ by performing a sequence of transformations $(a_q, b_q, c_q) \mapsto (a_q, b_q, c'_q)$, where*

$$c'_q := q^{-1} [3]_q a_q b_q - c_q, \quad (3)$$

combined with permutations of a_q, b_q , and c_q .

Theorem 1 means that every Markov triple (a, b, c) , has a unique q -analogue (a_q, b_q, c_q) solution to (2), that we call *the q -Markov triple*. Furthermore, combined with Proposition 2 and under the assumption that the Markov conjecture holds true, Theorem 1 implies that every Markov number m has a unique q -analogue that we denote by m_q .

The first q -deformations of the Markov numbers are as follows

$$\begin{aligned} 1_q &= 1, \\ 2_q &= q + q^{-1}, \\ 5_q &= q^2 + q + 1 + q^{-1} + q^{-2}, \\ 13_q &= q^3 + 2q^2 + 2q + 3 + 2q^{-1} + 2q^{-2} + q^{-3}, \\ 29_q &= q^4 + 2q^3 + 4q^2 + 5q + 5 + 5q^{-1} + 4q^{-2} + 2q^{-3} + q^{-4}, \\ 34_q &= q^4 + 3q^3 + 4q^2 + 6q + 6 + 6q^{-1} + 4q^{-2} + 3q^{-3} + q^{-4}, \\ 89_q &= q^5 + 4q^4 + 7q^3 + 11q^2 + 14q + 15 + 14q^{-1} + 11q^{-2} + 7q^{-3} + 4q^{-4} + q^{-5}, \\ 169_q &= q^6 + 3q^5 + 8q^4 + 14q^3 + 20q^2 + 25q + 27 + 25q^{-1} + 20q^{-2} + 14q^{-3} + 8q^{-4} + 3q^{-5} + q^{-6}, \\ 194_q &= q^6 + 4q^5 + 9q^4 + 16q^3 + 23q^2 + 29q + 30 + 29q^{-1} + 23q^{-2} + 16q^{-3} + 9q^{-4} + 4q^{-5} + q^{-6}, \\ \dots &\quad \dots \end{aligned}$$

The sequence of coefficients of these polynomials does not appear in the OEIS. The study of combinatorial and geometric properties of this sequence seems to be an interesting and challenging open problem. In Section 5, we give one such combinatorial interpretation.

It is well-known that the classical Markov numbers are closely related to the modular group $\text{PSL}(2, \mathbb{Z})$. The q -Markov numbers we consider are related to q -deformed rational numbers [25] and the q -deformed modular group [22]. This connection is explored in Section 4. In particular, we calculate explicitly q -analogues of the Cohn matrices, following Aigner's classification [1], and prove (see Theorem 12) that the q -Markov numbers do not depend on the choice of the initial Cohn matrices.

Recall that every Markov number is associated with a so-called *snake graph*. For explanation of this notion; see, e.g. [33, 1, 35] and Section 2. A famous result states that Markov numbers are equal to the number of perfect matchings (or dimer coverings) of the corresponding snake graphs. For more details; see [33, Thm 7.1], and also [1, Thm. 7.12], [35, Thm. 5.1]. It is a general rule that the coefficients of q -deformed combinatorial objects have the same combinatorial meaning as the initial object, but more refined. We give a combinatorial model for q -Markov numbers in terms of the perfect matchings of snake graphs with weighted edges.

Our rule to assign weights to the edges of snake graphs is the following. The edges at the western and southern borders have weights q^{-1} and q alternately, all other edges have weight 1; see Section 5. Taking the product of weights of the edges of every perfect matching, we define the *weighted number* of perfect matchings which is a Laurent polynomial in q .

The main result of the paper is as follows (for a more precise statement; see Theorem 16).

Theorem 3. *The q -Markov number m_q equals the weighted number of perfect matchings in the snake graph corresponding to m .*

Figure 1 provides an example of a snake graph with weighted edges. The total number of perfect matchings of this graph is equal to the Markov number 433. Taking the product of weights of the edges of every perfect matching, one obtains the polynomial 433_q .

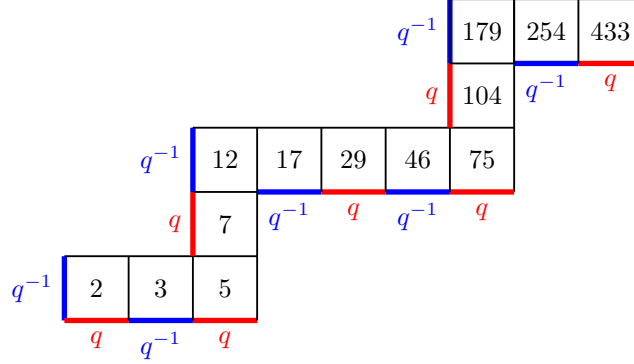


Figure 1: A snake graph with weighted edges. Edges coloured in red have weight q and edges coloured in blue have weight q^{-1} , other edges have weight 1.

Perfect matchings (or dimer coverings) with weighted edges are often used in combinatorics and other branches of mathematics and statistical physics. Ideas that are quite close to our approach were applied in [19] in the case of the Aztec diamond graph. For a survey; see, e.g. [17]. It could be an interesting question to ask whether the weight system corresponding to q -Markov numbers has some particular properties. Let us also mention that weighted perfect matchings of snake graphs were considered in [7], but with weights that are not defined on the edges.

Recall that Markov numbers are naturally presented in a form of a binary tree, and consequently every Markov number corresponds to a rational $t \in \mathbb{Q}$ such that $0 \leq t \leq 1$ and is often denoted by m^t . We refer to [1] for the details; see also Section 2. The claim that m^t is uniquely determined by its rational label t is equivalent to the Markov uniqueness conjecture, see [1].

As an application of Theorem 3, we will show in Section 5.5 that in the q -deformed case the map

$$t \in \mathbb{Q}_{[0,1]} \mapsto m_q^t$$

is injective, i.e. every q -Markov number m_q^t is uniquely determined by its rational label t . Note that a similar result, but for a different notion of q -deformed Markov numbers was obtained in [21].

Corollary 4. *Two q -Markov numbers, m_q^t and $m_q^{t'}$, with $t \neq t'$ are different Laurent polynomials in q .*

In fact, we will calculate t explicitly in terms of the coefficients of the Laurent polynomial m_q^t (Lemma 17).

Another application of Theorem 3 concerns the general properties of the Laurent polynomial m_q^t . The above examples suggest that the Laurent polynomials arising as q -Markov numbers enjoy several nice general properties. The following statement will be proved in Section 5.4.

Corollary 5. *The q -Markov numbers are Laurent polynomials with the following properties*

- (i) *the coefficients of every polynomial are non-negative integers;*
- (ii) *every such polynomial is monic, i.e. its leading order coefficient equals 1;*
- (iii) *every polynomial representing a q -Markov number is palindromic, or reciprocal, i.e.*

$$m_{q^{-1}}^t = m_q^t.$$

Combining with our results, the results of [31, 30] imply yet another property of q -Markov numbers.

Proposition 6. *The sequence of coefficients of every q -Markov number m_q^t , except for $2_q = q + q^{-1}$, is unimodal.*

Let us also add some historical comments on the subject. The recent notion of q -deformed rationals [25] and the related notion of q -deformed modular group [22] suggest that Markov numbers should have a natural q -deformation. Different variants of this notion were tackled in [20, 21, 18, 30], all of them based on the choice of a pair of q -deformed Cohn matrices. In [20, 21] a version of q -deformed Markov numbers was defined as upper right elements of some special q -deformed Cohn matrices. In [18, 30] they were defined as the traces divided by $[3]_q$ of particular q -deformed Cohn matrices.

The notion of q -Markov numbers we study in this paper coincides (up to a multiple) with that of [18, 30] and it is different from the one considered in [20, 21]. The first appearance of the equation (2) is due to Kogiso [18], who considered this equation as a property of the q -Markov numbers. The properties of palindromicity and unimodality of the q -Markov numbers were first obtained by Kantarcı Oğuz [30].

One important distinction between our approach and the previous ones is that we do not fix specific Cohn matrices to define q -Markov numbers, we define them and study them as solutions of the q -deformed Markov equation (2).

The paper is organized as follows. In Section 2, we present well-known results about Markov numbers. This allows us to introduce the notation and the main ideas of the theory. Section 3 contains the proof of the fact that every Markov number m^t has a canonical q -analogue m_q^t . In Section 4, we introduce the q -deformed action of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ and calculate q -deformed Cohn matrices. In Section 5, we define weighted snake graphs and prove our main result. Finally, in Section 6, we collect a number of observations, conjectures and open questions. In particular, we try to outline possible relations of q -Markov numbers to cluster algebra.

2 A compendium on Markov numbers

In this section, we collect well-known facts about the classical Markov numbers that will be relevant for this paper.

2.1 A recurrent procedure

Markov proved that all positive integer solutions of the equation (1) can be obtained from the simplest Markov triple $(a, b, c) = (1, 1, 1)$ by performing transformations $(a, b, c) \mapsto (a, b, c')$, where

$$c' := 3ab - c = \frac{a^2 + b^2}{c}, \quad (4)$$

combined with permutations of a, b , and c . These transformations are often called “mutations” because of the connection to cluster algebra; see Section 6.3. Note that both expressions in (4) are important, the first implies that c' remains an integer, while the second equality ensures that c' is positive. An elementary proof of this statement can be found in [1, p.46].

2.2 The Markov tree

The Markov numbers can be organized with the help of an infinite binary tree, sometimes called the Conway topograph; see [6]. The edges of the infinite binary tree have an orientation so that at each vertex there are one ingoing and two outgoing arrows. In our illustrations some edges are coloured in green indicating that there is a choice for their orientation but the choice does not affect the upper part of the tree, see Section 6.1 for further discussions.

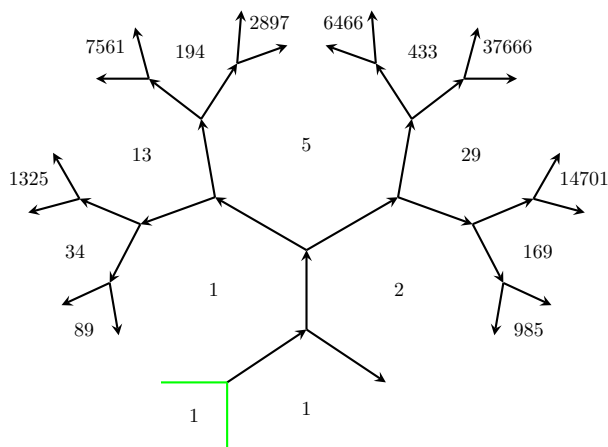


Figure 2: Markov numbers represented on the Conway topograph.

The tree is drawn in the plane cutting it into infinitely many regions, and every region is labelled by a Markov number. Three numbers around a vertex constitute a Markov triple, and the transformations (4) correspond to the following “fishbone branchings”:

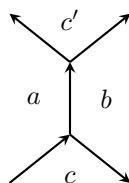


Figure 3: The fishbone

The tree of Markov numbers can be compared with the yet more ancient and better known Farey tree; see, e.g. [16, 1, 34].

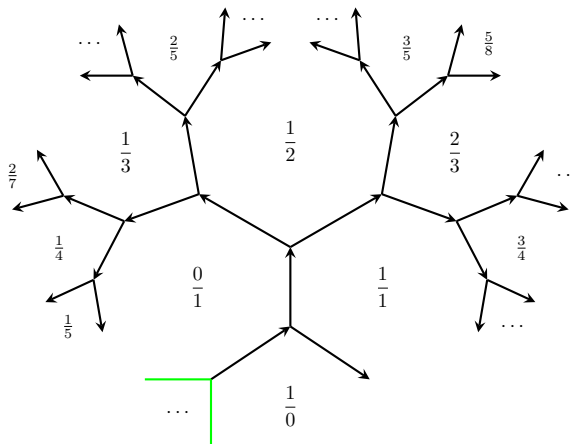


Figure 4: The Farey tree of rational numbers.

Rational numbers label regions bounded by the binary tree, and obey the following local rule: every

two rationals, $\frac{r}{s}$ and $\frac{r'}{s'}$, separated by an edge produce a third rational

$$\frac{r}{s} \oplus \frac{r'}{s'} := \frac{r + r'}{s + s'}$$

called the median, or the Farey sum of $\frac{r}{s}$ and $\frac{r'}{s'}$.

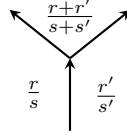


Figure 5: The Farey sum.

It is known that the Farey tree contains all rational numbers $\mathbb{Q} \cup \{\frac{1}{0}\}$, and every rational appears exactly once. Identifying the trees, one then equips every Markov number with a rational label. In other words, one has a map

$$t \in \mathbb{Q}_{[0,1]} \mapsto m^t$$

widely believed to be injective, although this is the Markov uniqueness conjecture; see [1].

2.3 Cohn matrices

There exists an amazing way of computing Markov numbers using 2×2 unimodular integer matrices. It was discovered by Harvey Cohn [10]. An element $C \in \text{SL}(2, \mathbb{Z})$

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1$$

is called a *Cohn matrix* if b is a Markov number and $\text{Tr}(C) = a + d = 3b$.

A classification of Cohn matrices is given in [1, Thm. 4.8], all such matrices are parametrized by one integer parameter $n \in \mathbb{Z}$ and can be obtained by multiplying two initial matrices $A(n)$ and $B(n)$

$$A(n) = \begin{pmatrix} n & 1 \\ 3n - n^2 - 1 & 3 - n \end{pmatrix}, \quad B(n) = \begin{pmatrix} 2n + 1 & 2 \\ -2n^2 + 4n + 2 & 5 - 2n \end{pmatrix}. \quad (5)$$

Observe that

$$B(n) = A(n)A(n+1).$$

The construction is again based on a binary tree:

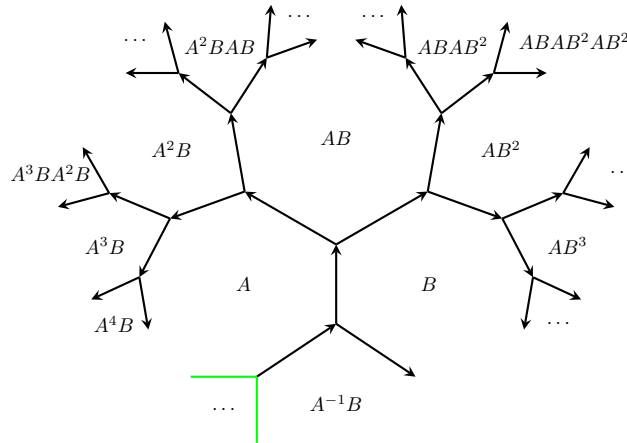


Figure 6: The Cohn tree.

The following statement is due to Harvey Cohn [10], its elegant proof can be found in [1].

Once again, identifying the binary trees, every matrix in the Cohn tree can be labelled by a rational number. The notation due to Aigner is C_t . The tree is determined by $C_{\frac{0}{1}} = A(n)$ and $C_{\frac{1}{1}} = B(n)$, for some fixed n , that are called *initial Cohn matrices*.

Theorem 7 ([10, 1]). *Every matrix C_t appearing in this tree is a Cohn matrix, its upper right element is precisely the corresponding Markov number m^t , while its trace equals $3m^t$.*

The most common choice of the Cohn matrices (5) corresponds to $n = 0, 1$, or 2:

$$A(0) = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}, \quad B(0) = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix},$$

$$A(1) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad B(1) = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix},$$

$$A(2) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B(2) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Remark. Markov numbers appear simultaneously as traces and as upper right elements of Cohn matrices. However, the notion of trace of a matrix is much more fundamental than its particular entry. This obvious reasoning becomes more important in the q -deformed case. We will see that the traces of q -deformed Cohn matrices are independent of their choice, while the upper right elements are different.

2.4 Snake graphs

A *snake graph* is a particular connected graph in the plane that has only vertical and horizontal edges. It can be obtained by gluing elementary square boxes along edges. A new box is attached to the previous one at the right or at the top.

Snake graphs related to Markov numbers appear in Harvey Cohn's paper [11, Fig 1.] and in Propp's paper [33]. Choosing a rational number $t = \frac{k}{s} \in [0, 1]$, one draws in the plane a line of slope t and one picks all of the 1×1 squares in the \mathbb{Z}^2 -grid crossed by the diagonal of the $k \times s$ rectangle. The resulting snake graph is called in [11] the “polygonal approximation” of the diagonal of the $k \times s$ rectangle. We reproduce here (see Fig. 7 left) a somewhat transposed figure from Cohn's paper (see Fig. 1 in [11]) for $t = \frac{3}{5}$.

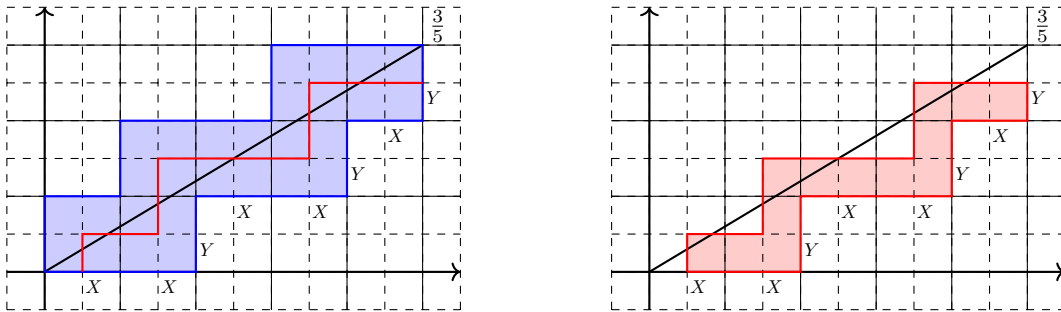


Figure 7: Snake graphs. The Cohn snake (left) vs the domino snake \mathcal{G}_t (right).

The construction of the snake graph that will be relevant for us refines that of Cohn. Tracing the “median zig-zag line” joining the point $(\frac{1}{2}, 0)$ and $(s, k - \frac{1}{2})$ (see the red line on Fig. 7, left) and collecting the $\frac{1}{2} \times \frac{1}{2}$ square boxes under the line, one obtains the snake graph that we denote by \mathcal{G}_t . Note that \mathcal{G}_t consists of $\frac{1}{2} \times \frac{1}{2}$ boxes and thus contains more boxes than Cohn's polygonal approximation. The snake

graph $\mathcal{G}_{\frac{3}{5}}$ in Fig. 7 (right) is precisely that of our previous example depicted in Fig. 1. The snake graph \mathcal{G}_t is called the “domino graph” in [1].

Let us mention that snake graphs are useful in several branches of combinatorics connected to continued fractions, cluster algebras, and many other topics; see [28, 9] and references therein. A systematic study of arbitrary snake graphs was initiated by Canakci and Schiffler in [8]. For a survey; see [35].

Recall that a *perfect matching* of a graph is a subset of its edges such that every vertex of the graph is incident to exactly one edge of the matching. Perfect matchings in simplest snake graphs are collected in Fig. 8 .



Figure 8: Examples of perfect matchings. There are two perfect matchings for a single box and three for a two-box snake.

The following statement is a well-known fundamental result explaining the combinatorial nature of Markov numbers.

Theorem 8. *Every Markov number m^t is equal to the number of perfect matchings in \mathcal{G}_t .*

This result first appears in [33, Thm. 7.1]. A self-contained transparent proof can be found in [1, Thm. 7.12]. The statement can be deduced from the result of [9] about continued fractions. Note that the result of [9] is actually more general. Another approach based on connections with cluster algebras is outlined in [35].

The main goal of this paper is to give a generalization of Theorem 8. To this end we will assign weights to the edges of \mathcal{G}_t .

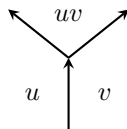
2.5 Christoffel words

The lower boundary of the snake graphs gives rise to a Christoffel word. This notion plays an important role in the theory of Markov numbers. We refer to [34] for details.

Fix a rational number $t \in [0, 1]$ and consider the associated Cohn’s snake graphs defined in the previous section. Along the lower boundary of the graph, assign the letter X to each horizontal step of length 1 and the letter Y to each vertical step of length 1. One can do the same on the lower boundary of \mathcal{G}_t except that in \mathcal{G}_t the initial horizontal step and the final vertical step are of length $\frac{1}{2}$. The *Christoffel word* associated to t , denoted by w_t , is the word in the letters X, Y read from left to right along the lower boundary of the snake graph.

Example 1. From Fig 7 one obtains $w_{\frac{3}{5}} = XYXYXYXY$.

Christoffel words label regions of the binary tree according to the following local rule



where u and v are words in X, Y and uv is the concatenated word. The binary tree of Christoffel words is as follows.

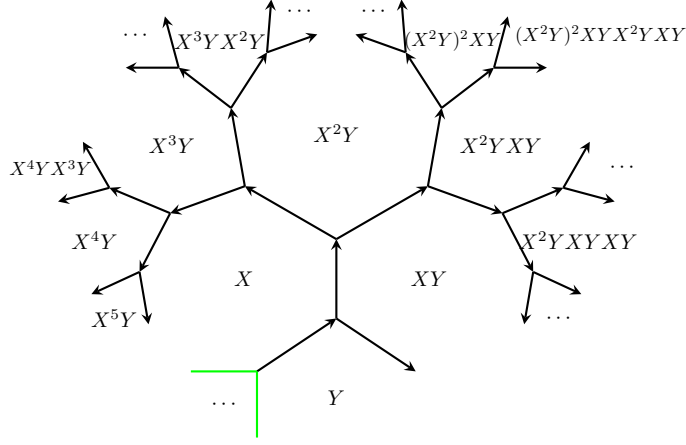


Figure 9: The tree of Christoffel words in X, Y

Let us collect here some important facts about Christoffel words.

1. Every Christoffel word is of the form $w = X\pi Y$ where π is a palindromic word in X, Y ; see [34, Thm 2.3.1].
2. The rational number corresponding to w_t is given by the simple formula:

$$t = \frac{\#Y\text{'s in } w_t}{\#X\text{'s in } w_t}.$$

3. Substituting the initial Cohn matrices $A(n)$ and $A(n+1)$ in place of X and Y in a Christoffel word w_t leads to the Cohn matrix

$$C_t(n) = w_t(A(n), A(n+1)).$$

The corresponding Markov number is given by

$$m^t = \frac{1}{3} \text{Tr } M_t(n).$$

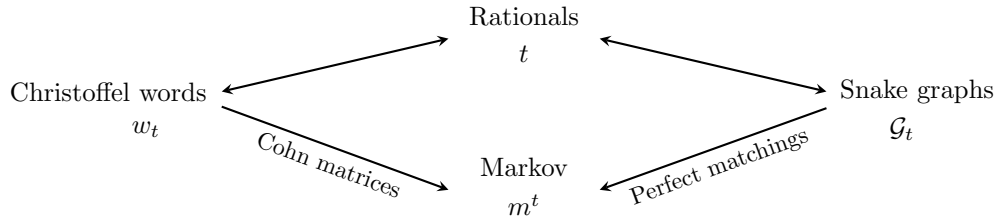
This is a reformulation of Theorem 7.

Remark. It is often convenient to substitute $A = X$ and $B = XY$ in a Christoffel word w . Doing so, one obtains a Christoffel word in the letter A and B , that we denote by \tilde{w} . For instance, for $t = \frac{3}{5}$ one has $\tilde{w} = ABABBB$. In terms of matrices

$$C_t(n) = w_t(A(n), A(n+1)) = \tilde{w}_t(A(n), B(n)).$$

2.6 Correspondences

In the previous sections, one has described the following correspondences:



We will describe q -deformations of Cohn matrices and perfect matchings that lead to polynomials m_q^t .

3 Proof of the uniqueness result

In this section we prove Theorem 1 stating that every triple of Markov numbers (a, b, c) has a unique q -deformation (a_q, b_q, c_q) .

3.1 Proof of Proposition 2

We start with the recurrent procedure described in Proposition 2. Recall that $(a_q, b_q, c_q) \in \mathbb{Z}[q^{\pm 1}]^3$ is called a q -Markov triple, if this is a triple of Laurent polynomials with integer coefficients, which are solutions to (2), and

$$(a, b, c) := (a_q, b_q, c_q)|_{q=1}$$

is a Markov triple of integers.

Lemma 9. *If $(a_q, b_q, c_q) \in \mathbb{Z}[q^{\pm 1}]^3$ is a q -Markov triple, then so is the triple (a_q, b_q, c'_q) with c'_q given by (3).*

Proof. It is a simple direct computation to check that if (a_q, b_q, c_q) satisfies (2), then (a_q, b_q, c'_q) is also a solution to (2). As in the classical Markov case, this is just a reformulation of the Vieta formulas for a quadratic equation. In addition, evaluating at $q = 1$, one obtains

$$c' = 3ab - c = \frac{a^2 + b^2}{c} > 0,$$

so that $(a_q, b_q, c'_q)|_{q=1} = (a, b, c')$ is a Markov triple. \square

We want to show that performing a sequence of transformations of the type (3) on an arbitrary q -Markov triple (a_q, b_q, c_q) one can eventually arrive at the triple $(1, 1, 1)$.

Denote by d_1, d_2, d_3 the degrees of the polynomials a_q, b_q, c_q respectively, and by $\alpha_1, \alpha_2, \alpha_3$ the leading coefficients of the polynomials a_q, b_q, c_q respectively. We assume $d_1 \leq d_2 \leq d_3$ and proceed by induction on d_3 .

Case $d_1 = d_2 = d_3 = 0$. In this case $(a_q, b_q, c_q) = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_{>0}^3$ is a Markov triple satisfying in addition the relation

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = (q + 1 + q^{-1}) \alpha_1 \alpha_2 \alpha_3 + 2 - q - q^{-1}.$$

This immediately implies $\alpha_1 \alpha_2 \alpha_3 = 1$ so that $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$ is the only triple of (positive) constant polynomials satisfying (2).

Case $d_3 \geq 1$. Since (a_q, b_q, c_q) satisfies (2), one obtains the following relations on the degrees and leading coefficients:

$$2d_3 = d_1 + d_2 + d_3 + 1, \quad \alpha_3^2 = \alpha_1 \alpha_2 \alpha_3.$$

In the expression of c'_q the term $q^{-1} [3]_q a_q b_q$ has degree $d_1 + d_2 + 1$ and leading coefficient $\alpha_1 \alpha_2$. The above relations imply that $q^{-1} [3]_q a_q b_q$ has degree d_3 and the leading coefficient α_3 , the same as c_q . Hence as the higher order term in $c'_q = q^{-1} [3]_q a_q b_q - c_q$ vanishes, and therefore c'_q is of degree d'_3 , which is strictly less than d_3 .

Repeating successively transformations on the polynomial of the triple with greatest degree will eventually lead to a triple of polynomials of degree 0, which as seen above can only be $(1, 1, 1)$. Observing that the transformation (3) is an involution one gets Proposition 2.

3.2 End of the proof

Finally Theorem 1 readily follows from Proposition 2, the fact that the transformation (3) is an involution, together with the statement that a Markov triple (a, b, c) appears only once in the Markov tree. For the latter; see [1, Thm 3.3].

4 q -Markov numbers from the modular group

In this section, we present the q -deformed version of the computations with Cohn matrices, that we explained in Section 2.3. The q -deformed Cohn matrices we consider belong to the q -deformed version of $\mathrm{PSL}(2, \mathbb{Z})$ described in [22]. As explained in [27], this q -deformation is the projective version of the classical Burau representation.

Note that a specific choice of q -deformed Cohn matrices was considered in [22, 21, 18, 30]. We prove that the resulting q -deformed Markov numbers do not depend on the choice of Cohn matrices.

4.1 q -deformed action of $\mathrm{PSL}(2, \mathbb{Z})$

Recall that the group $\mathrm{PSL}(2, \mathbb{Z})$ is generated by any two of the following three elements

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad L = TST = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad (6)$$

All three of these standard generators will be useful for explicit calculations. The elements T and S satisfy the relations $S^2 = (TS)^3 = 1$.

Following [25, 22], consider the following matrices

$$T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \quad S_q = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}, \quad L_q = T_q S_q T_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}. \quad (7)$$

depending on the parameter q . They correspond to linear-fractional transformations acting on the space $\mathbb{Z}(q)$ of rational functions in q with integer coefficients, that (slightly abusing the notation) we also denote by T_q and S_q :

$$T_q(f(q)) = qf(q) + 1, \quad S_q(f(q)) = -\frac{1}{qf(q)}, \quad L_q(f(q)) = -\frac{qf(q)}{qf(q) + 1}. \quad (8)$$

It is very easy to check that these operators satisfy exactly the same relations as the classical generators of $\mathrm{PSL}(2, \mathbb{Z})$, namely

$$S_q^2 = (T_q S_q)^3 = 1.$$

Therefore, they generate a $\mathrm{PSL}(2, \mathbb{Z})$ -action on $\mathbb{Z}(q)$.

One arrives at the following conclusion that will be important for the sequel. Consider the action of $\mathrm{PSL}(2, \mathbb{Z})$ generated by (8). Every element of $\mathrm{PSL}(2, \mathbb{Z})$ corresponds to a linear-fractional transformation on $\mathbb{Z}(q)$ and can be represented by a 2×2 matrix with coefficients that are polynomial in q , defined modulo a scalar multiple, which is a power of q . To make the choice of a 2×2 matrix representing an element of $\mathrm{PSL}(2, \mathbb{Z})$ canonical, it suffices to fix the determinant of the matrix. In our case, it will always be possible to choose the determinant equal to 1.

Remark. Consider the matrices

$$T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \quad L_q^{-1} = \begin{pmatrix} q^{-1} & 0 \\ -1 & 1 \end{pmatrix}. \quad (9)$$

Set $\tilde{L}_q^{-1} := qL_q^{-1}$, one then can check the following braid relation $T_q \tilde{L}_q^{-1} T_q = \tilde{L}_q^{-1} T_q \tilde{L}_q^{-1}$. This implies that the matrices T_q and \tilde{L}_q generate a representation

$$\rho_3 : B_3 \rightarrow \mathrm{GL}(2, \mathbb{Z}(q)), \quad \sigma_1 \mapsto T_q, \quad \sigma_2 \mapsto \tilde{L}_q^{-1}$$

of the braid group B_3 by 2×2 matrices with coefficients that are rational functions in q . Here σ_1 and σ_2 are the standard generators of B_3 . This representation has been known for almost a hundred years under the name of (reduced) Burau representation; see [5].

4.2 q -deformed Cohn matrices

Consider the Cohn matrices $A(n)$ and $B(n)$, as in (5). Our next goal is to calculate their q -deformations $A(n)_q$ and $B(n)_q$. Since these matrices are defined up to a scalar multiple, as are all of the elements of $\text{PSL}(2, \mathbb{Z})$, it is natural to look for q -deformations satisfying the additional condition

$$\det(A(n)_q) = \det(B(n)_q) = 1. \quad (10)$$

This fixes a canonical choice of the matrices representing elements of $\text{PSL}(2, \mathbb{Z})$.

We will use the following standard notation for q -integers: $[n]_q := \frac{q^n - 1}{q - 1}$; for positive n , one has $[n]_q = q^{n-1} + q^{n-2} + \dots + 1$.

We have the following computational statement.

Proposition 10. *The q -deformation of the matrix $A(n)$ satisfying the condition (10) is as follows:*

$$A(n)_q = \begin{pmatrix} q^{2-n}[n]_q & q^{1-n} \\ [n]_q[3-n]_q - q^{n-1} & q^{-1}[3-n]_q \end{pmatrix}. \quad (11)$$

Proof. First check that $A(n) = L^{3-n} S L^n$, where L and S are given by (6). The q -deformation of $A(n)$ is then given by

$$L_q^{3-n} S_q L_q^n,$$

where L_q and S_q are as in (7). One has

$$\begin{aligned} L_q^{3-n} S_q L_q^n &= \begin{pmatrix} q^{3-n} & 0 \\ q[3-n]_q & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \begin{pmatrix} q^n & 0 \\ q[n]_q & 1 \end{pmatrix} \\ &= \begin{pmatrix} q^{4-n}[n]_q & q^{3-n} \\ q^2[n]_q[3-n]_q - q^{n+1} & q[3-n]_q \end{pmatrix}. \end{aligned}$$

and we divide through by q^2 so that the determinant is 1. Finally one gets $A(n)_q = q^{-2} L_q^{3-n} S_q L_q^n$. \square

We will also need the companion Cohn matrix $B(n)_q$, where $B(n)$ is given by (5).

Corollary 11. *The q -deformation of the matrix $B(n)$ satisfying the condition (10)*

$$B(n)_q = \begin{pmatrix} q^{1-n}[n+1]_q[2]_q - q & q^{-n}[2]_q \\ q^{-1}[n+1]_q[3-n]_q[2]_q - [n+1]_q - q^{n-1}[3-n]_q & q^{-2}[3-n]_q[2]_q - q^{-1} \end{pmatrix}. \quad (12)$$

Proof. $B(n)$ is given by (5) can be expressed in terms of $A(n)$. Indeed, $B(n) = A(n)A(n+1)$, and therefore

$$B(n)_q = A(n)_q A(n+1)_q.$$

After a simple computation, one deduces (12) from (11). \square

Example 2. The following special choices

$$\begin{aligned} A(0)_q &= \begin{pmatrix} 0 & q \\ -q^{-1} & q^{-1}[3]_q \end{pmatrix}, & B(0)_q &= \begin{pmatrix} q^2 & [2]_q \\ q[2]_q & q^{-2}[4]_q + 1 \end{pmatrix}, \\ A(1)_q &= \begin{pmatrix} q & 1 \\ q & q^{-1}[2]_q \end{pmatrix}, & B(1)_q &= \begin{pmatrix} [3]_q & q^{-1}[2]_q \\ q^{-1}[4]_q & q^{-2}[3]_q \end{pmatrix}, \\ A(2)_q &= \begin{pmatrix} [2]_q & q^{-1} \\ 1 & q^{-1} \end{pmatrix}, & B(2)_q &= \begin{pmatrix} q^{-1}[4]_q + 1 & q^{-2}[2]_q \\ q^{-1}[2]_q & q^{-2} \end{pmatrix} \end{aligned}$$

are particularly useful in practice and will be needed in Section 5.

4.3 q -Markov numbers as traces of Cohn matrices

Consider now the Cohn tree, as depicted in Fig. 6, and replace the classical matrices $A(n), B(n)$ by the q -deformed matrices $A(n)_q, B(n)_q$.

The main statement of this section is the following.

Theorem 12. *For every matrix C_t of the Cohn tree, the trace of the q -deformed matrix is equal to $q^{-1} [3]_q m_q^t$, where m_q^t is the q -deformed analogue of m^t defined by (2).*

Proof. We proceed by induction on the position of the matrices in the Cohn tree.

(1) The induction basis. From (11), one has

$$\begin{aligned} \text{Tr}(A(n)_q) &= q^{2-n} [n]_q + q^{-1} [3-n]_q \\ &= \frac{1}{q-1} \left(q^{2-n} (q^n - 1) + q^{-1} (q^{3-n} - 1) \right) \\ &= \frac{1}{q-1} \left(q^2 - q^{2-n} + q^{2-n} - q^{-1} \right) \\ &= \frac{q^3 - 1}{q(q-1)} \\ &= q^{-1} [3]_q. \end{aligned}$$

It is independent of n and is equal to $q^{-1} [3]_q = q + 1 + q^{-1}$, as required.

A similar computation using (12) gives

$$\text{Tr}(B(n)_q) = q^{-1} [3]_q (q + q^{-1}),$$

in accordance with the second q -Markov number $2_q = q + q^{-1}$.

Note also that from the relation $B(n) = A(n)A(n+1)$ one gets

$$A(n)_q B(n)_q^{-1} = A(n)_q A(n+1)_q^{-1} A(n)_q^{-1},$$

and therefore the trace of this matrix is equal to $\text{Tr}(A(n+1)_q)$:

$$\text{Tr} \left(A(n)_q B(n)_q^{-1} \right) = q + 1 + q^{-1}.$$

(2) The induction step. Suppose that q -deformed Cohn matrices M and N are separated by some branch of the Cohn tree

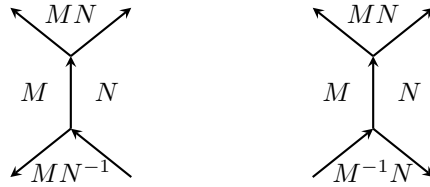


Figure 10: Cohn branchings.

and we know that

$$\text{Tr}(M) = q^{-1} [3]_q m^t, \quad \text{Tr}(N) = q^{-1} [3]_q m^{t'}, \quad \text{Tr}(MN^{-1}) = q^{-1} [3]_q m^{t \ominus t'}, \quad (13)$$

where $t \ominus t'$ is the operation opposite to the Farey sum $t \oplus t'$:

$$\frac{r}{s} \ominus \frac{r'}{s'} := \frac{r - r'}{s - s'}.$$

We need to prove that $\text{Tr}(MN) = q^{-1} [3]_q m^{t \oplus t'}$. We give the details in the case Fig. 10 (left), the second case is similar.

We use the well-known formula for 2×2 matrices

$$\text{Tr}(MN) = \text{Tr}(M)\text{Tr}(N) - \text{Tr}(MN^{-1}). \quad (14)$$

Then (13) gives

$$\text{Tr}(MN) = q^{-1} [3]_q \left(q^{-1} [3]_q m^t m^{t'} - m^{t \oplus t'} \right).$$

The right-hand side of this equation is nothing but the Vieta transform (3).

Therefore $\text{Tr}(MN)$ is a multiple of $q^{-1} [3]_q$, and after division by this Laurent polynomial it satisfies (2) (cf. Proposition 2). It then follows from Theorem 1 that $\frac{\text{Tr}(MN)}{q+1+q^{-1}}$ is a q -Markov number corresponding to the Markov number $m^{t \oplus t'}$. Hence the induction step.

Theorem 12 is proved. \square

Remark. Theorem 12 means in particular that the traces of Cohn matrices do not depend on the integer parameter n , and these traces are always a multiple of $q^{-1} [3]_q$. Observe also that the upper right elements heavily depend on the choice of initial Cohn matrices. Note that the definition of q -deformed Markov numbers in [22, 21] is based on a choice of Cohn matrices and makes use of the upper right element of it.

4.4 Entries of Cohn matrices

It turns out that there are interesting relations between the trace and the entries of the first row and the second column of a q -deformed Cohn matrix, as suggested in [18, Lemma 2.3].

Lemma 13. *Let*

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

be an arbitrary q -deformed Cohn matrix. One then has the following relations

$$\frac{\text{Tr}(C)}{q^{-1} [3]_q} = \left(q^{-1} - q^{-2} \right) c_{11} + q^{-1} c_{12}, \quad (15)$$

$$= q^2 c_{12} + q(1 - q) c_{22}. \quad (16)$$

Proof. Let us give the details for the relation (15), the proof of (16) is similar.

The proof goes along the same lines as that of Theorem 12. We proceed by induction on the Cohn tree.

(1) The induction basis. A simple direct computation shows that the relation (15) holds for the Cohn matrices $A(n)_q$ and $B(n)_q$ in the initial triple.

(2) The induction step. Let $(M, N, M^{-1}N)$ be a triple of Cohn matrices around a vertex of the tree. We assume that the relation (15) holds for $C = M, N$, and $M^{-1}N$, as in Fig. 10 (right). Note that the case Fig. 10 (left) is similar. We want to show that the relation holds also for $C = MN$.

Consider the coefficients of the matrices

$$M = (m_{ij}), \quad N = (n_{ij}), \quad P = M^{-1}N = (p_{ij}), \quad C = MN = (c_{ij}),$$

and for simplicity set

$$\alpha := q^{-1} - q^{-2}, \quad \beta := q^{-1}.$$

The proof of (15) actually does not depend on the choice of constants α and β .

The relation (14) reads

$$\begin{aligned} \frac{\text{Tr}(MN)}{q^{-1} [3]_q} &= q^{-1} [3]_q \left(\frac{\text{Tr}(M)}{q^{-1} [3]_q} \frac{\text{Tr}(N)}{q^{-1} [3]_q} \right) - \frac{\text{Tr}(P)}{q^{-1} [3]_q} \\ &= (m_{11} + m_{22})(\alpha n_{11} + \beta n_{12}) - (\alpha p_{11} + \beta p_{12}), \end{aligned}$$

with $P = M^{-1}N$ so

$$p_{11} = m_{22}n_{11} - m_{12}n_{21} \quad \text{and} \quad p_{12} = m_{22}n_{12} - m_{12}n_{22}.$$

One then computes for $C = MN$:

$$\begin{aligned} \frac{\text{Tr}(C)}{q^{-1}[3]_q} &= (m_{11} + m_{22})(\alpha n_{11} + \beta n_{12}) - (\alpha(m_{22}n_{11} - m_{12}n_{21}) + \beta(m_{22}n_{12} - m_{12}n_{22})), \\ &= \alpha(m_{11}n_{11} + m_{12}n_{21}) + \beta(m_{11}n_{12} + m_{12}n_{22}) \\ &= \alpha c_{11} + \beta c_{12}. \end{aligned}$$

Hence the lemma. \square

Remark. It is worth mentioning that the expression (16) coincides with the numerator of the “left q -deformation” of the rational $\frac{c_{12}}{c_{22}}$; see [3]. The same expression appears in [25] as the Jones polynomial corresponding to the rational knot parametrized by $\frac{c_{12}}{c_{22}}$.

5 A combinatorial model

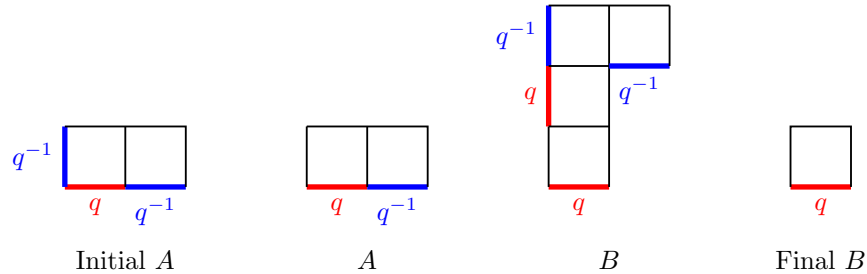
In this section, we prove Theorem 3 stating that q -Markov numbers count perfect matchings of snake graphs with weighted edges.

5.1 Snake graphs with weighted edges

Definition 14. The *weighted snake graph* $\mathcal{G}_t(q)$ is obtained from the snake graph \mathcal{G}_t by assigning weights on the edges of \mathcal{G}_t according to the following rule.

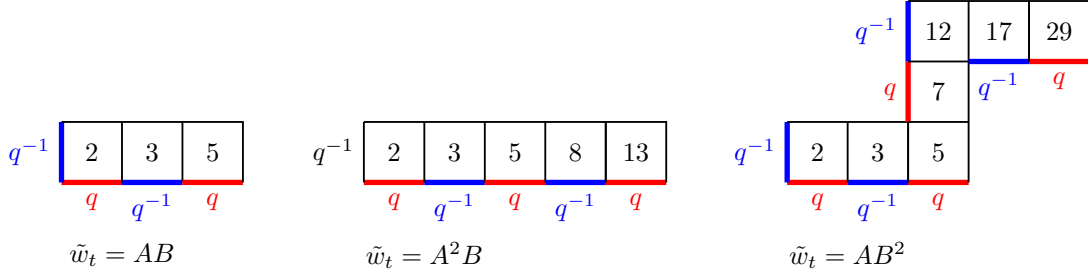
1. Vertical edges on the Western boundary of \mathcal{G}_t are assigned with weights q^{-1}, q, \dots alternating from bottom to top.
2. Horizontal edges on the Southern boundary of \mathcal{G}_t are assigned with weights q, q^{-1}, \dots alternating from left to right.
3. Other edges of \mathcal{G}_t are assigned with weight 1.

Alternatively, for $0 < t < 1$, the graph $\mathcal{G}_t(q)$ can be constructed directly from the Christoffel words. More precisely we consider the Christoffel word $\tilde{w}_t(A, B)$ obtained from the word w_t by substituting the letters $A = X$, $B = XY$ (see Section 2.5). Each letter in \tilde{w}_t gives rise to a piece of snake graph according to the following description:



All the pieces are connected by gluing the rightmost vertical edge of the previous piece to the bottom leftmost vertical edge of the next piece.

Example 3. (a) Examples of $\mathcal{G}_t(q)$, for $t = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, corresponding to the Markov numbers 5, 13, 29, respectively.



(b) Figure 1 gives an example for $t = \frac{3}{5}$.

5.2 Detailed statement of the main theorem

Let us first recall the standard definitions.

Definition 15. (i) Given a graph \mathcal{G} with weighted edges, we will use the standard notion of the *weighted perfect matching* in \mathcal{G} . For a given perfect matching \mathbf{m} , we define its weight, $\text{wt}(\mathbf{m})$, as the product of the weights of its edges. The weight is a power of q .

(ii) Given a snake graph $\mathcal{G}_t(q)$ with weighted edges, we define the *weighted number of perfect matchings* as the generating function for the weighted perfect matchings:

$$\mu^t(q) := \sum_{\mathbf{m}} \text{wt}(\mathbf{m}).$$

Other terms sometimes used in the dimer theory are “statistics” or “statistical sum”.

The following statement is a more detailed reformulation of our main result announced in the introduction as Theorem 3.

Theorem 16. *The q -deformed Markov number m_q^t is equal to the weighted number of perfect matchings $\mu^t(q)$ in the snake graph $\mathcal{G}_t(q)$.*

Our proof, given in the next section, will follow that of Theorem 7.12 of [1].

5.3 Proof of the main theorem

Fix a rational $t \in]0, 1[$ and the corresponding Christoffel word

$$\tilde{w}_t = Aw_1 \cdots w_s B,$$

with letters $w_i \in \{A, B\}$. Let n be the number of boxes in the snake graph $\mathcal{G}_t(q)$. We count the boxes travelling along the snake from left to right and bottom to top.

For each $i \in \{1, \dots, n\}$, let us denote by μ_i the weighted number of perfect matchings of the partial snake graph containing the first i boxes. It is convenient to label the i -th box with the value μ_i . In particular, for the initial snake graph one has

with

$$\mu_1 = q + q^{-1}, \quad \mu_2 = q + q^{-1} + q^{-2}. \quad (17)$$

Here μ_1 is easily obtained by enumerating the 2 perfect matchings on a single box and μ_2 is obtained by enumerating the 3 perfect matchings on the graph with two boxes (see example of Fig. 8).

We can compute recursively μ_i in the snake graph $\mathcal{G}_t(q)$, looking at how it evolves through the addition of a type A piece or a type B piece of graph (see Section 5.1 and [1], p.146).

Attachment of an A -piece

α_1	α_2	β_1	β_2
q	q^{-1}	q	q^{-1}

Then

$$\beta_1 = \alpha_2 + q\alpha_1, \quad \beta_2 = \beta_1 + q^{-1}\alpha_2,$$

which can be rewritten as

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} q & 1 \\ q & 1 + q^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Observe that the matrix $\begin{pmatrix} q & 1 \\ q & 1+q^{-1} \end{pmatrix}$ is nothing but the q -deformed Cohn matrix $A(1)_q$, already calculated in Section 4.2.

Attachment of a B -piece (except for the final one)

One easily obtains the following relations between the weighted number of perfect matchings:

$$\begin{aligned}\gamma_1 &= \alpha_2 + q\alpha_1 \\ \gamma_2 &= \gamma_1 + q^2\alpha_1 \\ \beta_1 &= \gamma_2 + q^{-1}\gamma_1 \\ \beta_2 &= \beta_1 + q^{-2}\gamma_1\end{aligned}$$

from which one deduces

$$\begin{aligned}\beta_1 &= (q^2 + q + 1)\alpha_1 + (1 + q^{-1})\alpha_2 \\ \beta_2 &= (q^2 + q + 1 + q^{-1})\alpha_1 + (1 + q^{-1} + q^{-2})\alpha_2\end{aligned}$$

which again can be rewritten as

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = B(1)_q \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

where $B(1)_q$ is the q -deformed Cohn Matrix companion to $A(1)_q$.

Attachment of the final B -piece

Diagram illustrating a 3-cell 2D chain with boundary conditions. The cells are labeled α_1 , α_2 , and μ_n . The bottom boundary has red segments under α_1 and μ_n , and a blue segment under α_2 . The red segments are labeled q , and the blue segment is labeled q^{-1} .

In this case, we have

$$\mu_n = (q \quad 1) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

We conclude that the weighted number of perfect matchings of $\mathcal{G}_t(q)$ is given by

$$\mu_n = (q \ 1) M_s \cdots M_1 \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},$$

where for $k \in \{1, \dots, s\}$,

$$M_k = \begin{cases} A(1)_q, & \text{if } w_k = A, \\ B(1)_q, & \text{if } w_k = B. \end{cases}$$

and where μ_1 and μ_2 are given by (17).

Observing now that

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = B(1)_q \begin{pmatrix} q^{-1} - q^{-2} \\ q^{-1} \end{pmatrix}, \quad \text{and} \quad (q \ 1) = (1 \ 0) A(1)_q,$$

and using the fact that $w_1 \cdots w_s$ is a palindrome, we get

$$\mu_n = (1 \ 0) A(1)_q M_1 \cdots M_s B(1)_q \begin{pmatrix} q^{-1} - q^{-2} \\ q^{-1} \end{pmatrix}.$$

The matrix

$$C_t = A(1)_q M_1 \cdots M_s B(1)_q =: (m_{ij})$$

is precisely the Cohn q -matrix corresponding to the Markov number m_q^t , therefore

$$\mu_n = (1 \ 0) M_t \begin{pmatrix} q^{-1} - q^{-2} \\ q^{-1} \end{pmatrix} = (q^{-1} - q^{-2} \quad q^{-1}) \begin{pmatrix} m_{11} \\ m_{12} \end{pmatrix} = (q^{-1} - q^{-2}) m_{11} + q^{-1} m_{12}.$$

Applying Lemma 13, we finally get

$$\mu^t(q) = \mu_n = \frac{\text{Tr}(C_t)}{q^{-1}[3]_q} = m_q^t,$$

as desired.

5.4 Properties of the q -Markov polynomials

In this section, we prove Corollary 5 and Proposition 6.

We start by proving Corollary 5.

Part (i). Given a q -Markov number m_q^t , that is, a Laurent polynomial in q , it follows from Theorem 16, that every coefficient of this polynomial counts perfect matchings with some fixed weight. Therefore this coefficient is non-negative.

Part (ii). For every snake graph \mathcal{G}_t corresponding to a Markov number m^t , there exists a unique perfect matching of maximal (or minimal) weight. Indeed, choosing all edges of weight q (red edges), it is easy to see that there exists a unique way to complete them as a perfect matching. Therefore, the Laurent polynomial m_q^t has highest (and lowest) order coefficient 1.

Part (iii). The palindromic property $m_{q^{-1}}^t = m_q^t$ follows directly from (3) and Proposition 2. Indeed, transformations (3) do not affect the palindromic property, which obviously holds for the initial triple.

Now we prove Proposition 6.

It was proved in [30, Thm. 7.3] that $\text{Tr}(C_t)/[3]_q$ is given by the rank polynomial of a circular fence poset, and these polynomials were proved to be unimodal in [31, Corollary 7.4], except in some exceptional cases. In the case of Cohn matrices the only possible exception corresponds to the matrix $C_{\frac{1}{1}}$, i.e., the q -Markov number 2_q .

5.5 The weighted uniqueness conjecture

Let us now prove Corollary 4. The proof consists of calculating an explicit formula to recover the rational number corresponding to a q -Markov number.

Lemma 17. *Let $m_q^t = q^d + \alpha q^{d-1} + \dots + \alpha q^{1-d} + q^{-d}$ be a q -Markov number. The corresponding rational label is expressed in terms of the degree of m_q^t and its coefficient of the term of degree $d - 1$:*

$$t = \frac{d - \alpha}{\alpha + 1}.$$

Proof. There exists exactly one perfect matching of \mathcal{G}_t with weight q^{-d} , namely “the minimal matching” consisting of all boundary edges of weight q^{-1} completed (in a unique way) with boundary edges of weight 1. One has:

$$d = \# \text{boxes in Cohn's snake},$$

that is, the number of letters in the Christoffel word decreased by 1, i.e. $d = (\#X + \#Y\text{'s in } w_t) - 1$.

We know, by palindromicity, that α is also the coefficient of the term of degree $1 - d$, hence the number of perfect matching of \mathcal{G}_t of weight q^{1-d} . Such a perfect matching is obtained by taking all edges of weight q^{-1} , but one. This is possible only by removing in the minimal matching a horizontal edge of weight q^{-1} with the opposite boundary edge sharing the same box, and replacing these two by the two internal vertical edges of the box. There are $(\#X\text{'s} - 1)$ horizontal edges of weight q^{-1} , therefore $\alpha = (\#X\text{'s in } w_t) - 1$.

Finally one gets

$$t = \frac{\#Y\text{'s in } w_t}{\#X\text{'s in } w_t} = \frac{d - \alpha}{\alpha + 1}.$$

Hence the lemma. □

Corollary 4 follows.

6 Further directions

We finish this article with miscellaneous observations that did not fit into the main core but constitute subjects for future research.

6.1 Through the looking glass: sixing of the tree

Our first observation concerns the classical Markov numbers and Cohn matrices.

The usual way to represent the Markov numbers in the form of a tree uses a binary tree growing in one direction; see Fig. 2. This tree starts from the Markov triple $(1, 2, 5)$. However, the Markov numbers tree can be extended infinitely in all directions, which leads to what is effectively 6 copies of the Markov numbers tree, starting instead from the initial triple $(1, 1, 1)$. We find this “sixing” of the tree quite relevant, because it better illustrates the role of Cohn matrices and extends Aigner’s classification of them [1].

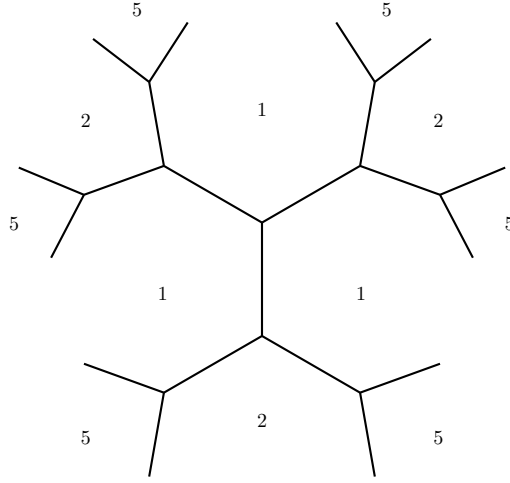


Figure 11: Markov numbers on the Conway topograph, in all directions.

In order to replicate this in the Cohn matrix case, we need to “glue” together copies of the tree. The result of this procedure leads to the following “complete Cohn tree”, where $A(n)$ and $B(n)$ are as in (5).

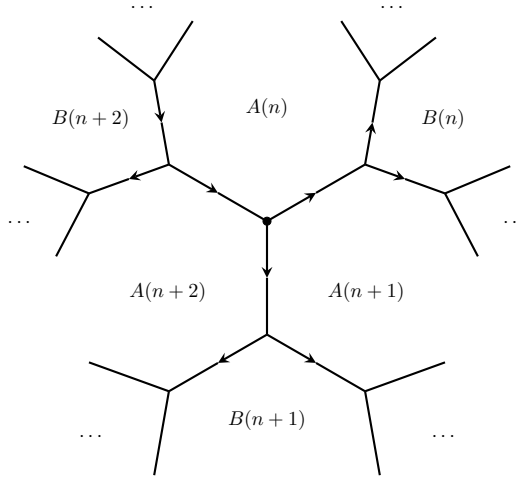
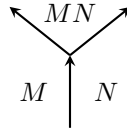


Figure 12: Cohn tree in all directions.

The tree is oriented, at every vertex of the tree there are exactly *one ingoing and two outgoing* edges. Without this condition, the construction is not consistent. The rule for constructing the tree and calculating the matrices in it is as in the usual case:



One can prove the following.

1. The initial triple of Cohn matrices $A(n)$, $A(n+1)$, $A(n+2)$ is the only possible, so that the tree at Fig. 12 is the most general one.

2. This complete Cohn tree is consistent for any choice of orientation.

The proof goes along the lines of the proof of Aigner's [1] Theorem 4.7, with only a minor modification, and we do not go into the details.

We observe from the tree at Fig. 12 that besides the thoroughly studied pair of Cohn matrices $(A(n), B(n))$, several other pairs naturally appear. For instance, the "inverted pair" $(B(n), A(n+1))$ which is not a part of Aigner's classification and some other interesting cases, according to the choice of orientation.

6.2 Combinatorial models for q -rationals

In a Cohn matrix the Markov number appears twice, as the top right entry and as the third of the trace. Therefore with the q -deformed Cohn matrix one obtains two q -deformations of the Markov number. As we have seen in this article the q -deformation given by the trace is independent of the choice of the initial Cohn matrix. This is not the case of the top right entry as one can immediately see in the examples of matrices $A(n)_q$ and $B(n)_q$ in Section 4.2 that the entries depend on n .

Fixing $n = 1$, we will denote by \tilde{m}_q^t the top right entry of the Cohn matrix M_t obtained from the initial $A(1)_q$ and $B(1)_q$. The proof of Theorem 16 can be easily adapted in order to obtain a combinatorial model for \tilde{m}_q^t .

More precisely, defining $\tilde{\mathcal{G}}_t(q)$ as the weighted graph obtained from $\mathcal{G}_t(q)$ by removing the weight q of the first horizontal edge, one obtains:

Proposition 18. *The q -deformed Markov number \tilde{m}_q^t is the weighted number of perfect matchings of $\tilde{\mathcal{G}}_t(q)$.*

Example 4. For $t = \frac{2}{3}$, the top right entry of the matrix $A(1)_q B(1)_q^2$ is

$$\tilde{m}_q^t = q^3 + 3q^2 + 5q + 6 + 6q^{-1} + 5q^{-2} + 2q^{-3} + q^{-4}$$

and coincides with the weighted number of perfect matchings of the graph below.

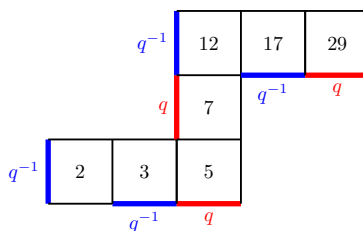


Figure 13: Weighted graph $\tilde{\mathcal{G}}_t(q)$ for $t = \frac{2}{3}$.

More generally, in the q -deformed matrices of $\text{PSL}(2, \mathbb{Z})$ the top right entry corresponds to a numerator of a q -rational [25]. The above observation leads to another combinatorial model for the q -rationals. The details will be developed elsewhere.

6.3 F -polynomials in the theory of cluster algebras

The theory of cluster algebras by Fomin and Zelevinsky gives an excellent framework to study Markov numbers and the Markov equation. Various work on Markov numbers and (generalized) Markov equation has been produced in this framework, see e.g. [2, 4, 12, 23, 29, 33].

Here we introduce a 3-parameter deformation of Cohn matrices related to the F -polynomials of the Markov cluster algebra.

First we introduce some ingredients of the theory of cluster algebras without going deeply into details. We refer to [13, 14] for an overview and the main definitions.

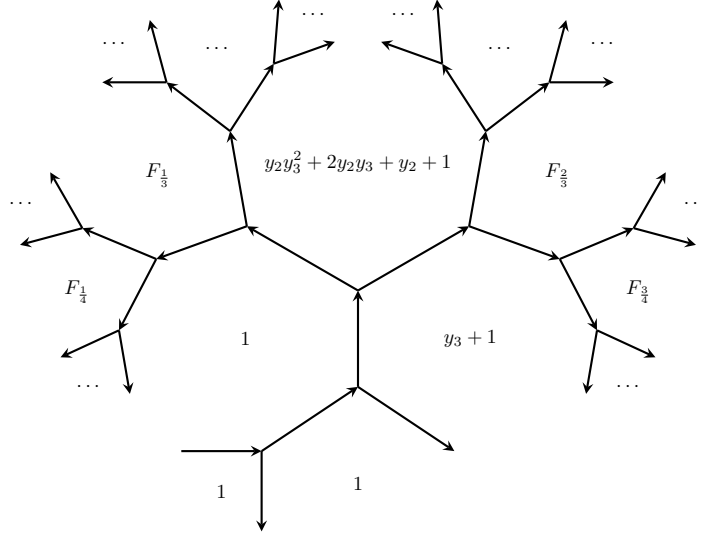


Figure 16: Tree of F -polynomials.

Example 5. (a) Further examples of Markov cluster variables are as follows.

$$\begin{aligned}
\mathcal{X}_{\frac{1}{3}} &= \frac{x_1^6 + 3x_1^4 x_2^2 + 3x_1^2 x_2^4 + x_2^6 + (2x_1^4 + 2x_1^2 x_2^2)x_3^2 + x_1^2 x_3^4}{x_2^2 x_3^3}, \\
\mathcal{X}_{\frac{1}{4}} &= \frac{(x_1^8 + 4x_1^6 x_2^2 + 6x_1^4 x_2^4 + 4x_1^2 x_2^6 + x_2^8 + (3x_1^6 + 6x_1^4 x_2^2 + 3x_1^2 x_2^4)x_3^2 + (3x_1^4 + 2x_1^2 x_2^2)x_3^4 + x_1^2 x_3^6)}{x_2^3 x_3^4}, \\
\mathcal{X}_{\frac{2}{3}} &= \frac{x_1^8 + 4x_1^6 x_2^2 + 6x_1^4 x_2^4 + 4x_1^2 x_2^6 + x_2^8 + (2x_1^6 + 5x_1^4 x_2^2 + 4x_1^2 x_2^4 + x_2^6)x_3^2 + x_1^4 x_3^4}{x_1 x_2^2 x_3^4}, \\
\mathcal{X}_{\frac{3}{4}} &= \frac{x_1^{12} + 6x_1^{10} x_2^2 + 15x_1^8 x_2^4 + 20x_1^6 x_2^6 + 15x_1^4 x_2^8 + 6x_1^2 x_2^{10} + x_2^{12}}{x_1^2 x_2^3 x_3^6} \\
&\quad + \frac{(3x_1^{10} + 14x_1^8 x_2^2 + 26x_1^6 x_2^4 + 24x_1^4 x_2^6 + 11x_1^2 x_2^8 + 2x_2^{10})x_3^2}{x_1^2 x_2^3 x_3^6} \\
&\quad + \frac{(3x_1^8 + 8x_1^6 x_2^2 + 8x_1^4 x_2^4 + 4x_1^2 x_2^6 + x_2^8)x_3^4 + x_1^6 x_3^6}{x_1^2 x_2^3 x_3^6}.
\end{aligned}$$

(b) The corresponding F -polynomials are the following

$$\begin{aligned}
F_{\frac{1}{3}} &= y_2^2 y_3^3 + 3y_2^2 y_3^2 + 3y_2^2 y_3 + y_2^2 + 2y_2 y_3 + 2y_2 + 1, \\
F_{\frac{1}{4}} &= y_2^3 y_3^4 + 4y_2^3 y_3^3 + 6y_2^3 y_3^2 + 4y_2^3 y_3 + 3y_2^2 y_3^2 + y_2^3 + 6y_2^2 y_3 + 3y_2^2 + 2y_2 y_3 + 3y_2 + 1, \\
F_{\frac{2}{3}} &= y_1 y_2^2 y_3^4 + 2y_1 y_2^2 y_3^3 + y_2^2 y_3^4 + y_1 y_2^2 y_3^2 + 4y_2^2 y_3^3 + 6y_2^2 y_3^2 + 4y_2^2 y_3 + 2y_2 y_3^2 + y_2^2 + 4y_2 y_3 + 2y_2 + 1, \\
F_{\frac{3}{4}} &= y_1^2 y_2^3 y_3^6 + 2y_1^2 y_2^3 y_3^5 + 2y_1 y_2^3 y_3^6 + y_1^2 y_2^3 y_3^4 + 8y_1 y_2^3 y_3^5 + y_2^3 y_3^6 + 12y_1 y_2^3 y_3^4 + 6y_2^3 y_3^5 + 8y_1 y_2^3 y_3^3 \\
&\quad + 2y_1 y_2^2 y_3^4 + 15y_2^3 y_3^4 + 2y_1 y_2^3 y_3^2 + 4y_1 y_2^2 y_3^3 + 20y_2^3 y_3^3 + 3y_2^2 y_3^4 + 2y_1 y_2^2 y_3^2 + 15y_2^3 y_3^2 \\
&\quad + 12y_2^2 y_3^3 + 6y_2^3 y_3 + 18y_2^2 y_3^2 + y_2^3 + 12y_2^2 y_3 + 3y_2 y_3^2 + 3y_2^2 + 6y_2 y_3 + 3y_2 + 1
\end{aligned}$$

As recalled in the previous sections the Markov number m^t appears as the top right element of the corresponding Cohn matrix. We introduce a deformation of the Cohn matrices using the parameters

(y_1, y_2, y_3) leading to a generalization of this fact. Consider the 3-parameter quantisation governed by

$$\begin{aligned}\hat{A} &= \begin{pmatrix} y_3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2 & 0 \\ y_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} y_2 + y_2 y_3 & 1 \\ y_2 & 1 \end{pmatrix}. \\ \hat{B} &= \begin{pmatrix} y_3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2 & 0 \\ y_2 & 1 \end{pmatrix} \begin{pmatrix} y_3 & 0 \\ y_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} y_1 y_2 y_3^2 + y_2 y_3^2 + 2 y_2 y_3 + y_2 & y_3 + 1 \\ y_2 y_3 + y_2 & 1 \end{pmatrix}.\end{aligned}$$

Note that these are product of what we can consider as y -deformed versions of two standard generators T and L of $\mathrm{SL}_2(\mathbb{Z})$. Specializing $y_1 = y_2 = y_3 = q$ in \hat{A}, \hat{B} will return the q -deformed Cohn matrices $A(2)_q, B(2)_q$, up to a power of q .

Conjecture 19. *For every rational t , corresponding to the Christoffel word $\tilde{w}_t(A, B)$, the deformed matrix \hat{C}_t given by $\tilde{w}_t(\hat{A}, \hat{B})$ has a top right entry equal to the F -polynomial F_t , up to a unit element of the ring $\mathbb{Z}[y_1, y_2, y_3]$. In particular for $0 \leq t \leq 1$ the top right entry is exactly F_t .*

Computer experiments are the evidence for this conjecture. For instance

$$\begin{aligned}\hat{C}_{\frac{1}{2}} = \hat{A}\hat{B} &= \begin{pmatrix} (y_1 y_2 y_3^2 + y_2 y_3^2 + 2 y_2 y_3 + y_2 + 1) y_2 (y_3 + 1) & y_2 y_3^2 + 2 y_2 y_3 + y_2 + 1 \\ (y_1 y_2 y_3^2 + y_2 y_3^2 + 2 y_2 y_3 + y_2 + y_3 + 1) y_2 & y_2 y_3 + y_2 + 1 \end{pmatrix}, \\ \hat{C}_{\frac{1}{6}} = \hat{A}^{-1} \hat{B} &= \begin{pmatrix} y_1 y_3 + y_3 + 1 & y_2^{-1} \\ -y_1 y_2 y_3 & 0 \end{pmatrix}.\end{aligned}$$

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