ON TANGENT CONES OF SCHUBERT VARIETIES

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ABSTRACT. We consider tangent cones of Schubert varieties in the complete flag variety. We investigate the problem when the tangent cones of two different Schubert varieties coincide. In particular, we show that all Schubert varieties corresponding to the Coxeter elements of the Weyl group have the same tangent cone.

1. INTRODUCTION

Let $\mathcal{F}$ be the algebraic variety of all complete flags in $\mathbb{C}^n$. Recall that a complete flag $F \in \mathcal{F}$ is an increasing sequence of subspaces

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n, \qquad \dim V_k = k.$$

Choosing the standard basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of $\mathbb{C}^n$, one defines the standard flag, $F_0 \in \mathcal{F}$, for which $V_k = \mathbb{C}^k := \langle \varepsilon_1, \ldots \varepsilon_k \rangle$, for all $1 \le k \le n$. The group $\mathrm{GL}(n, \mathbb{C})$ of linear transformations of $\mathbb{C}^n$ transitively acts on $\mathcal{F}$. The Borel subgroup $B \subset \mathrm{GL}(n, \mathbb{C})$ of upper-triangular matrices is the stabilizer of the standard flag F_0, so $\mathcal{F} = \mathrm{GL}(n, \mathbb{C})/B$.

Let us recall some well-known facts. The group B acts naturally on $\mathcal{F}$ (by left multiplication). The variety $\mathcal{F}$ is a disjoint union of B-orbits called *Schubert cells*. Schubert cells are indeed cells of the most classical CW decomposition of $\mathcal{F}$. Schubert cells are parametrized by elements of the symmetric group S_n. Namely, the group S_n acts naturally in $\mathbb{C}^n$, and hence in $\mathcal{F}$, and for every $w \in S_n$, there exists a unique Schubert cell, which contains the w-image of the standard flag F_0. We denote this cell by $\mathcal{C}_w$. Its complex dimension is equal to the *length* of w, i.e., the minimal ℓ in a decomposition

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell},$$

where $s_i \in S_n$ are the elementary transpositions. The number of Schubert cells of complex dimension m is the coefficient at t^m in the polynomial

$$\prod_{k=1}^{n} (1 + t + \ldots + t^k).$$

In particular, there is a unique 0-dimensional cell, which is F_0 and a unique $\dfrac{n(n+1)}{2}$-dimensional cell, which is dense in $\mathcal{F}$.

The closure $\mathcal{X}_w$ of a Schubert cell $\mathcal{C}_w$ is called a Schubert variety. The Schubert variety $\mathcal{X}_w$ is the union of the Schubert cell $\mathcal{C}_w$ and all Schubert cells $\mathcal{C}_{w'}$ corresponding to permutations w' which precede w with respect to the natural partial ordering of S_n. In particular, every Schubert variety contains the point F_0.

With a Schubert variety $\mathcal{X}_w$, we associate two subsets of the tangent space $T_{F_0}\mathcal{F}$:

- the *tangent cone* $\mathcal{T}_w$, which is the set of vectors tangent to $\mathcal{X}_w$ at F_0;

- the Zariski *tangent space* $\mathcal{Z}_w$ which is *spanned* by $\mathcal{T}_w$.

The tangent cones $\mathcal{T}_w$ are algebraic subvarieties of $T_{F_0}\mathcal{F}$; they have the same dimensions as $\mathcal{X}_w$ (and $\mathcal{C}_w$). The tangent cone $\mathcal{T}_w$ and tangent space $\mathcal{Z}_w$ (as well as their dimensions) coincide if and only if F_0 is not a singular point of $\mathcal{X}_w$.

Certainly, the Schubert varieties $\mathcal{X}_w$ and $\mathcal{X}_{w'}$ coincide only when $w = w'$; however, the equalities $\mathcal{Z}_w = \mathcal{Z}_{w'}$ or $\mathcal{T}_w = \mathcal{T}_{w'}$ may occur for $w \neq w'$ (since the second implies the first, the first occurs "more often" than the second).

For the further discussion, let us introduce the most natural local coordinate system in a (Zariski) neighborhood of F_0 in $\mathcal{F}$. For a flag $\{V_k\}$ sufficiently "close" to F_0, there exists a unique "triangular" basis in $\mathbb{C}^n$,

$$v_1 = (1, x_{21}, \ldots, x_{n1}), \qquad v_2 = (0, 1, x_{32}, \ldots, x_{n2}), \quad \ldots \qquad v_n = (0, \ldots, 0, 1),$$

such that V_k is spanned by $v_1, \ldots, v_k$. The numbers $a_{ij}, i > j$ are coordinates of the flag $\{V_k\}$ (with $F_0 = (0, \ldots, 0)$); the same numbers may be regarded as coordinates in $T_{F_0}\mathcal{F}$. (This coordinate system provides a natural identification of $T_{F_0}\mathcal{F}$ with the space $\mathfrak{n}_-$ of strictly lower triangular matrices.) When n is not too large, we will use the more convenient notations $x_i = x_{i,i+1}, y_i = x_{i,i+2}$, etc.

Zariski tangent spaces $\mathcal{Z}_w$ were thoroughly studied, see [15, 12, 2] and references therein. The following result of Lakshmibai [12] provides an explicit description of $\mathcal{Z}_w$. The space $\mathcal{Z}_w$, viewed as a subspace of $\mathfrak{n}_-$, is the linear span of the elements e_α of the Chevalley basis, such that

$$\alpha \in R^+ \mid s_\alpha \geq w,$$

where $s_\alpha \in S_n$ is the reflection associated with α and $\geq$ is the Bruhat order. The above result, of course, answers the question, under which condition two different Schubert varieties $\mathcal{X}_w$ and $\mathcal{X}_{w'}$ have the same Zariski tangent space. On the contrary, the structure of tangent cones $\mathcal{T}_w$, although it has been an active area of research (see [2, 4, 5, 6, 3] and references therein), is not well understood, in particular, the problem of their coincidence is mostly open.

Let us consider some examples. If $n = 3$, then $\dim \mathcal{F} = 3$ and the local coordinates are x_1, x_2, y. There are 6 Schubert varieties of dimensions $0, 1, 1, 2, 2, 3$, and the middle four are:

$$\mathcal{X}_{(132)} = \{V_1 = \mathbb{C}^1\}, \quad \mathcal{X}_{(213)} = \{V_2 = \mathbb{C}^2\}, \quad \mathcal{X}_{(231)} = \{V_1 \subset \mathbb{C}^2\}, \quad \mathcal{X}_{(312)} = \{V_2 \supset \mathbb{C}^1\}.$$

In our local coordinates these are $x_1 = y = 0$, $x_2 = y = 0$, $y = 0$, $y = x_1 x_2$, respectively. We see that, within the domain of our coordinate system, $\mathcal{X}_{(231)}$ is the tangent plane (at the origin) to $\mathcal{X}_{(312)}$; thus $\mathcal{T}_{(231)} = \mathcal{T}_{(312)} = \mathcal{Z}_{(231)} = \mathcal{Z}_{(312)}$.

The first examples of singular Schubert varieties appear when $n = 4$. There are two of them:

$$\mathcal{X}_{(3412)} = \{V_1 \subset \mathbb{C}^3, \mathbb{C}^1 \subset V_3\} \quad \text{and} \quad \mathcal{X}_{(4231)} = \{V_2 \cap \mathbb{C}^2 \neq 0\}.$$

Our local coordinates in the 6-dimensional manifold $\mathcal{F}$ are $x_1, x_2, x_3, y_1, y_2, z$, the equations of the two Schubert varieties are $z = 0$, $y_1 x_3 + x_1 y_2 - x_1 x_2 x_3 = 0$ and $y_1 y_2 - z x_2 = 0$ and the tangent cones are, respectively, the cone $y_1 x_3 + x_1 y_2 = 0$ in the hyperplane $z = 0$ and the cone $y_1 y_2 - z x_2 = 0$. in the whole space $T_{F_0}\mathcal{F}$. It is not difficult to observe that the 24 Schubert varieties have 16 different tangent cones and 14 different tangent spaces.

For $n = 5$, we observe not only singular, but also reducible tangent cones (the Schubert varieties themselves are always irreducible). Moreover, different tangent cones can share components and even contain each other. The simplest example is provided by the 8-dimensional Schubert varieties

$$\mathcal{X}_{(35421)} = \{V_1 \subset \mathbb{C}^3\}, \quad \mathcal{X}_{(43521)} = \{V_2 \subset \mathbb{C}^4\} \quad \text{and} \quad \mathcal{X}_{(45231)} = \{V_1 \subset \mathbb{C}^4, \mathbb{C}^2 \cap V_3 \neq 0\}.$$

With respect to the local coordinates $x_1, x_2, x_3, x_4, y_1, y_2, y_3, z_1, z_2, t$, the first two varieties (and hence their tangent cones) are linear subspaces $z_1 = t = 0$ and $z_2 = t = 0$, while the third one is described by the equations $t = 0, \det \begin{bmatrix} y_1 & x_2 & 1 \\ z_1 & y_2 & x_3 \\ 0 & z_2 & y_3 \end{bmatrix} = 0$. This shows that the tangent cone $\mathcal{T}_{(45231)}$ is $\{t = z_1 z_2 = 0\}$, and this is the union $\mathcal{T}_{(35421)} \cup \mathcal{T}_{(435231)}$.

In this paper, we study the structure of the tangent cones $\mathcal{T}_w$ with the emphasis on the problem of their coincidence. Let us mention two cases when the coincidence of these tangent cones is known, or can be easily proved. The first one is the equality $\mathcal{T}_w = \mathcal{T}_{w^{-1}}$ which holds for every permutation w. This fact was conjectured (and checked for $n \leq 5$) in [6]; however, a short direct proof can be easily given, see Section 5.7. The second case is that of *Coxeter elements* of the permutation group. Recall that an element $w \in S_n$ is called a Coxeter element, if it is of length $n - 1$ and can be written in the form

$$w = s_{i_1} s_{i_2} \cdots s_{i_{n-1}}$$

in such a way that every transposition s_i, for $i = 1, 2, \ldots, n - 1$ enters the above product exactly once. The group S_n has 2^{n-2} different Coxeter elements. The Schubert varieties which correspond to the Coxeter elements of S_n have the same tangent cone; this follows from the fact that these Schubert varieties are non-singular at F_0, and hence their tangent cones coincide with their Zariski tangent spaces (see Section 3.4 for details). By the way, our example of coinciding tangent cones for $n = 3$ represents both cases: the permutations $(1, 3, 2)$ and $(3, 2, 1)$ are Coxeter elements inverse to each other. For $n = 4$, all pairs of permutations with equal tangent cones are either Coxeter, or inverse to each other. However, for $n = 5$, there appear pairs of non-inverse and non-Coxeter permutations with equal tangent cones; the first example of such a pair is $(13452), (13524)$.

We develop an efficient method to recognize when the tangent cones of two Schubert varieties coincide. The main ingredient of this method is the notion of a *pillar entry*. Every Schubert cell of the flag variety is determined by the $(n+1) \times (n+1)$ matrix of dimensions r_{ij} of the intersections $V_i \cap \mathbb{C}^j$ called the rank matrix; the corresponding Schubert variety is determined by inequalities $\dim(V_i \cap \mathbb{C}^j) \geq r_{i,j}$. For example, if $[r_{ij}]$ is the rank matrix corresponding to a permutation w, then the rank matrix corresponding to w^{-1} is obtained from $[r_{ij}]$ by a transposition. In Section 5.6, we prove that the whole matrix $[r_{ij}]$ is determined by a relatively small set of entries, which we call *pillar entries* (see Section 2.3 for a precise definition). Note that the notion of pillar entry is very close (yet different from) Fulton's notion of essential set [8], see the Appendix for a comparison.

We conjecture that if $\mathcal{T}_w = \mathcal{T}_{w'}$, then the pillar entries for w' are obtained from pillar entries for w by a *partial transposition*: if r'_{ij} is a pillar entry for w', then precisely the entries one of r_{ij}, r_{ji} is a pillar entry for w, and this entry is equal to r'_{ij} (and the same is true with the exchange $r \leftrightarrow r'$); see Section 2.4, Conjecture 2.11 for a precise statement. However,

the converse of this conjecture is false: examples show that a partial transposition of the set of pillar entries may lead to a set of entries which is not the set of pillar entries for any transposition, or is a set of pillar entries os a transposition of a different length. Some pillar entries are "linked," that is, they can be transposed or not transposed only simultaneously.

In Sections 3.2 and 3.3, we give some definition of a linkage, and hence of "admissible partial transposition"; our main result is Theorem 3.6, which states that an admissible partial transposition of pillars entries of w provides a set of pillar entries of some w', and that in this case $\mathcal{T}_w = \mathcal{T}_{w'}$. However, examples show that our definition of linkage is not sufficient: there are partial transpositions of pillar entries, which are not admissible in our sense, but which still preserve the tangent cone.

We also provide a number of examples and several enumerative results in small dimension and codimension. We were led by the numeric examples to the following "2^m-conjecture" (which is also closely related with the earlier mentioned conjecture): the number of Schubert varieties with an identical tangent cone is always a power of 2.

Let us mention that the problem of classification of tangent cones of Schubert varieties is closely related to the problem of classification of coadjoint orbits of the unitriangular group, see [11, 1] and the recent work [14]. As we already said, the tangent space to the flag variety is naturally identified with the nilpotent Lie algebra of lower-triangular matrices, and with the dual space of the Lie algebra of upper-triangular matrices:

$$T_{F_0}\mathcal{F} \simeq \mathfrak{n}_- \simeq \mathfrak{n}_+^*.$$

The B-action on $T_{F_0}\mathcal{F}$ then coincides with the coadjoint action. Every tangent cone $\mathcal{T}_w$ is B-invariant, as well as any irreducible component of $\mathcal{T}_w$; thus, it is a set of B-orbits. However, it is not true that B-orbits and irreducible components of tangent cones are the same thing. The first example which demonstrates this appears in S_6: the 6-dimensional tangent cone $\mathcal{T}_{(146252)}$ is a union of 5-dimensional B-orbits. We will not discuss this phenomenon in this paper.

2. BASIC NOTIONS

We recall the classical notion (and some properties) of rank matrix associated with two flags. Rank matrices provide a combinatorial way to characterize Schubert varieties and Schubert cells. Indeed, one of these flags will be chosen as the standard flag, so that the rank matrix coincides with the rank function of the corresponding permutation; see [9, 8]. We then define the notion of pillar entry of a rank matrix which is crucial for us.

We formulate our first conjecture that if two permutations, w and w', have identical tangent cones: $\mathcal{T}_w = \mathcal{T}_{w'}$, then the pillar entries of the corresponding rank matrices either coincide or transposed to each other.

2.1. **Rank matrix.** For any flag, the *rank matrix* is the $(n+1) \times (n+1)$ matrix $r = [r_{ij}]$ with the integer entries

$$r_{ij} = \dim V_i \cap \mathbb{C}^j, \qquad 0 \le i, j \le n.$$

The rank matrix is independent of the choice of a flag in a B-orbit. Moreover, it completely characterizes the corresponding B-orbit. More precisely, two different flags, $F \in C_w$ and $F' \in C_{w'}$, have the same rank matrix if and only if $w = w'$; see, e.g., [9]. We will denote by $r(w)$ the rank matrix corresponding to the Schubert cell C_w.

Obviously, one has:

$$r_{0k} = r_{k,0} = 0; \qquad r_{k,n} = r_{n,k} = k; \qquad r_{ij} + r_{i+1,j+1} \geq r_{i+1,j} + r_{i,j+1}.$$

The following statement is due to [8], see also [9] p.157. The Schubert cell C_w consists in flags such that the corresponding rank matrix is:

$$(1) \qquad\qquad r_{ij} = \#\{r \leq i \mid w(r) \leq j\}.$$

Example 2.1. The rank matrices $r(w)$ and $r(w^{-1})$ are transposed to each other. In this case, one has:

$$\mathcal{T}_w = \mathcal{T}_{w^{-1}}.$$

This statement was conjectured (and checked for $n \leq 5$) in [6]. However a short direct proof can be easily given, see Section 5.7.

2.2. **Permutation diagram.** The permutation $w \in S_n$ can be easily recovered from the rank matrix.

Definition 2.2. Given a permutation $w \in S_n$, the *diagram of* w is defined with the following convention. In an $(n+1) \times (n+1)$ grid, with row and columns numbered form 0 to n, we place a dot in the upper left corner of the cell with coordinates (i,j) whenever $j = w(i)$.

To make this visible, we usually put a $\bullet$ into the matrix, so that the permutation is encoded by the bullets.

Proposition 2.3. *If the rank matrix* $r(w)$ *is locally as follows:*

$$\begin{array}{c|c} a & a \\ \hline a & a+1 \end{array}$$

where $a+1$ *is the value in position* (i,j), *then the permutation* w *sends* i *to* j.

Proof. This readily follows from (1). $\qquad\square$

Example 2.4. Consider the case of dimension 4. a) The matrices

0	0	0	0	0
0	0	0	0	1
0	0	0	1	2
0	0	1	2	3
0	1	2	3	4

0	0	0	0	0
0	1	1	1	1
0	1	2	2	2
0	1	2	3	3
0	1	2	3	4

are the rank matrices corresponding to the longest element $w_0 = 4\,3\,2\,1$ and the identity element $w = 1\,2\,3\,4$ respectively.

b) The following matrices:

0	0	0	0	0
0	0	①	1	1
0	0	1	②	2
0	0	1	2	3
0	1	2	3	4

0	0	0	0	0
0	0	①	1	1
0	0	1	1	2
0	1	②	2	3
0	1	2	3	4

0	0	0	0	0
0	0	0	1	1
0	①	1	②	2
0	1	1	2	3
0	1	2	3	4

0	0	0	0	0
0	0	0	0	1
0	①	1	1	2
0	1	②	2	3
0	1	2	3	4

are the rank matrices corresponding to the four Coxeter elements in S_4:

$$s_1s_2s_3 = 2\,3\,4\,1, \qquad s_1s_3s_2 = 2\,4\,1\,3, \qquad s_2s_1s_3 = 3\,1\,4\,2, \qquad s_3s_2s_1 = 4\,1\,2\,3,$$

respectively. Note that the Schubert varieties of Coxeter elements are smooth at F_0.

c) Consider the elements $w_1 = 3\,4\,1\,2$ and $w_2 = 4\,2\,3\,1$ of S_4. The corresponding rank matrices are

0	0	0	0	0
0	0	0	①	1
0	0	0	1	2
0	①	1	2	3
0	1	2	3	4

and

0	0	0	0	0
0	0	0	0	1
0	0	①	1	2
0	0	1	2	3
0	1	2	3	4

The Schubert varieties $\mathcal{X}_{w_1}$ and $\mathcal{X}_{w_2}$ are the only singular Schubert varieties for $n = 4$, cf. [13].

The encircled entries will be later called "pillar", these entries determine the whole matrix, as explained in the next paragraph.

Example 2.5. For the maximal cell C_{w_0}, the rank matrix is given by:

$$r_{ij}(w_0) = \max\{0,\, i + j - n\}.$$

The smaller is the Schubert cell C_w, the bigger are the numbers $r_{ij}(w)$.

2.3. **The pillar entries.** It turns out that the rank matrix is completely determined by a few particular entries. The following notion is crucial for us.

Definition 2.6. An entry r_{ij} of a rank matrix $r(w)$ is called *pillar* if it satisfies the conditions

$$(2) \qquad \begin{cases} r_{ij} = r_{i-1\,j} + 1 = r_{i\,j-1} + 1, \\ r_{ij} = r_{i+1\,j} = r_{i\,j+1}. \end{cases}$$

In other words, the fragment of the rank matrix around a pillar entry is as follows:

$$\begin{array}{ccc} & a - 1 & \\ a - 1 & \text{ⓐ} & a \\ & a & \end{array}$$

We always encircle the pillar entries, in order to distinguish them.

In combinatorial terms, pillar entries can be characterized as follows. An entry r_{ij} of a rank matrix $r(w)$ is pillar if and only if

$$(3) \qquad \begin{cases} w(i) \le j, & w(i+1) > j, \\ w^{-1}(j) \le i, & w^{-1}(j+1) > i. \end{cases}$$

It is easy to see that these conditions are equivalent to (2).

Proposition 2.7. *Every Schubert cell is completely determined by the pillar entries of the rank matrix.*

We postpone the proof of the proposition to Section 5.6.

Remark 2.8. An algorithm that reconstructs the permutation w from the pillar entries of the rank matrix $r(w)$ will be presented in Section 4.2.

Example 2.9 (Pillars of the Coxeter elements)**.** It can be deduced from formula (3), that the rank matrix of every Coxeter element of S_n is determined by a sequence of $n-2$ inclusions:

$$V_i \subset \mathbb{C}^{i+1}, \qquad \text{or} \qquad \mathbb{C}^i \subset V_{i+1},$$

for $i \in \{1, \ldots, n-2\}$. The 2^{n-2} Coxeter elements correspond to an arbitrary choice of one of the above inclusions for every i. This gives $n-2$ pillar entries $r_{i,i+1} = i$ or $r_{i+1,i} = i$, for each $i \in \{1, 2, \ldots, n-2\}$ in the rank matrix. We will show in Section 3.4 that all Coxeter elements have the same tangent cone.

Remark 2.10. An entry r_{ij} of a rank matrix $r(w)$ is called *essential*, see [8] and also [7], if

(4)
$$\begin{cases} w(i) > j, & w(i+1) \le j, \\ w^{-1}(j) > i, & w^{-1}(j+1) \le i. \end{cases}$$

Equivalently, the rank matrix around an essential entry is as follows:

$$\begin{array}{ccc} & a & \\ a & \boxed{a} & a+1 \\ & a+1 & \end{array}$$

It is proved in [8] that every rank matrix (and therefore the corresponding Schubert variety) is completely characterized by its essential set.

The notions of essential and pillar entries are somewhat "complementary", as the inequality signs in formulas (3) and (4) are reversed, cf. Appendix for a comparison.

We believe that the notion of pillar entry (and its relation to Fulton's essential entries; see [8]) deserve a further study. In particular, the number of pillar entries for a given permutation is an interesting characteristic. Some of the basic properties of pillar entries will be presented in Section 4.

2.4. **The necessary condition: transposed pillars.** The following conjecture asserts that if two Schubert varieties have the same tangent cones, then they have the same number of pillars, whose values are also the same, and whose position in the respective rank matrices can only differ by transposition.

Conjecture 2.11. *Given two permutations, $w, w' \in S_n$, if $\mathcal{T}_w = \mathcal{T}_{w'}$ then the rank matrices $r(w)$ and $r(w')$ have the same number of pillar entries, and for every pillar entry r_{ij} of $r(w)$, one has the following alternative:*
a) the entry r'_{ij} of $r(w')$ is pillar and $r'_{ij} = r_{ij}$, or
b) the entry r'_{ji} of $r(w')$ is pillar and $r'_{ji} = r_{ij}$.

Examples 2.1 and 2.9 are the first examples that confirm our conjecture. We will give many other examples in the sequel.

2.5. **Restrictions: forbidden transpositions.** Note that the inverse of Conjecture 2.11 is false: two permutations with partially transposed pillar entries do not necessarily correspond to the same tangent cones.

0	0	0	0	0	0	0
0	0	0	0	①	1	1
0	0	0	0	1	②	2
0	0	0	0	1	2	3
0	0	0	1	2	3	4
0	0	1	2	3	4	5
0	1	2	3	4	5	6

0	0	0	0	0	0	0
0	0	0	0	0	1	1
0	0	0	0	1	②	2
0	0	0	0	1	2	3
0	①	1	1	2	3	4
0	1	1	2	3	4	5
0	1	2	3	4	5	6

Example 2.12. The simplest counterexample to the converse statement that we know is provided by the following permutations in S_6: $w = 456321$ and $w' = 564132$. Indeed, the corresponding rank matrices are:
respectively. The pillar entries are (partially) transposed, but the permutations have different length: $\ell(w) = 12$ and $\ell(w') = 11$, so that their tangent cones have different dimensions, and cannot coincide.

Note however the following interesting inclusion: $\mathcal{T}_{w'} \subset \mathcal{T}_w$.

Another restriction for partial transposition of pillars occurs more often than the above discussed one. Given a permutation w and the corresponding rank matrix $r(w)$, then a partial transposition of the pillar entries may not correspond to any rank matrix of any permutation.

Example 2.13. Consider the permutation $w = 3\,4\,5\,2\,1$ in S_5. The corresponding rank matrix is:

0	0	0	0	0	0
0	0	0	①	1	1
0	0	0	1	②	2
0	0	0	1	2	3
0	0	1	2	3	4
0	1	2	3	4	5

It turns out that there are no rank matrices with the following pillar entries:

a)

0	0	0	0	0	0
0			①		1
0					2
0					3
0		②			4
0	1	2	3	4	5

b)

0	0	0	0	0	0
0					1
0				②	2
0	①				3
0					4
0	1	2	3	4	5

Indeed, consider the case a). One has: $\mathbb{C}^1 \subset V_3$, and $V_2 \subset \mathbb{C}^4$. But, these conditions imply stronger ones: either $\mathbb{C}^1 \subset V_2$, or $V_3 \subset \mathbb{C}^4$. The corresponding rank matrices are

a1)

0	0	0	0	0	0
0	0	①	1	1	1
0	0	1	1	1	2
0	0	1	1	2	3
0	1	②	2	3	4
0	1	2	3	4	5

a2)

0	0	0	0	0	0
0	0	0	①	1	1
0	0	0	1	1	2
0	0	1	2	2	3
0	1	2	③	3	4
0	1	2	3	4	5

In both cases, a1) and a2), the corresponding permutations have length different from $\ell(w)$, so that their tangent cones are different from $\mathcal{T}_w$. The case b) is similar. In fact, the only other permutation w', such that $\mathcal{T}_w = \mathcal{T}_{w'}$ in the above example is $w' = w^{-1}$, that corresponds to the transposed rank matrix.

2.6. **More on partial transpositions.** Let us briefly discuss the partial transpositions of linked pillar entries. If one transpose some pillar entries of a rank matrix $r(w)$, but not all of them, then the following four possibilities may occur:

1) the resulting matrix is not a rank matrix of a permutation.

2) the resulting matrix is the rank matrix of some permutation w' that has a different set of pillar entries (cf. Example 2.13);

3) the resulting matrix is the rank matrix of a permutation w' that does have the given set of pillar entries, but of different length (cf. Example 2.12);

4) the "good case" where the resulting matrix is a rank matrix of a permutation that has the given set of pillar entries and the same tangent cone as w.

In view of Conjecture 2.11 and the above discussion, the main goal of this paper is to investigate which (partial) transpositions of pillar entries of a rank matrix $r(w)$ lead to a new permutation w' and do not change the tangent cone.

3. Admissible partial transpositions

In this section we describe classes of permutations in S_n with identical tangent cones. Given a permutation w, we define a series of operations called "admissible partial transpositions" and an equivalence class in S_n that consist of permutations related by such transpositions. We will prove that all permutations from such a class correspond to the same tangent cone.

However, the described classes are not maximal. Examples in the end of the section show that there are more permutations with identical tangent cones.

3.1. **Linked and dissociated pillar entries.** We define an equivalence relation on the set of pillar entries of a rank matrix. Roughly speaking, two pillar entries are in the same class if they are "close enough" to each other.

Definition 3.1. Given a permutation $w \in S_n$, and let r_{ij} and $r_{i'j'}$ be two pillar entries in the rank matrix $r_{ij}(w)$. These pillar entries are called *related* if the intervals:

$$[\min(i,j), \max(i,j)] \qquad \text{and} \qquad [\min(i',j'), \max(i',j')]$$
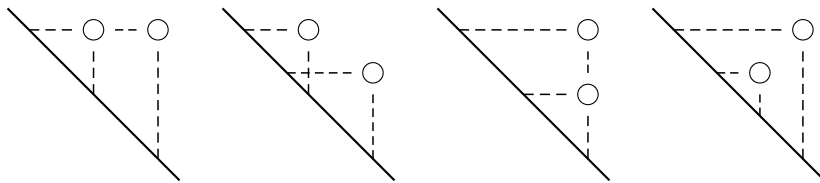
have a commun interior point.

FIGURE 1. Configurations for two related pillars

Otherwise the pillar entries are called *dissociated*.
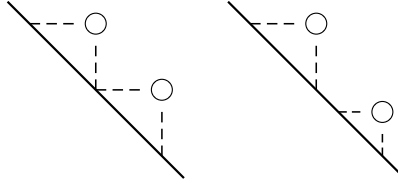
FIGURE 2. Configurations for two dissociated pillars

Figure 3.9 and Figure 2 in the case of 2 pillar entries present all possible configurations of related and dissociated pillar entries (up to their transpositions), respectively.

Example 3.2. The following rank matrix (in which we omit the extremal rows and columns) corresponding to the permutation $w = 12\,2\,9\,7\,6\,4\,10\,5\,3\,11\,1\,8 \in S_{12}$:

have three equivalence classes of linked pillar entries:

$$\{r_{22} = 1\}, \ \{r_{64} = 2\}, \ \{r_{67} = 4, \ r_{69} = 5, \ r_{97} = 6, \ r_{9,10} = 8, \ r_{11,7} = 7\}.$$

3.2. **The linking graph of pillar entries.** It is convenient to display the linking relations between the pillar entries using a graph.

Definition 3.3. The *linking graph* is defined as follows.

 (1) The set of vertices of the linking graph is the set of pillar entries of the rank matrix;
 (2) two vertices are connected by an edge whenever the corresponding pillar entries are related, cf. item (i) of Definition 3.1.

For instance, Example 3.2 corresponds to the following graph
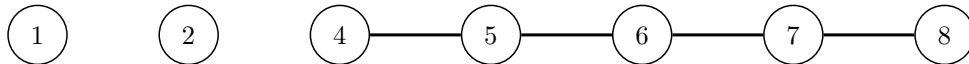
where we have labeled the vertices by the values of the pillar entries (omitting the positions as the values are all different).

The connected components of the linking graph corresponds to the equivalence classes.

3.3. **Admissible partial transpositions.** An *admissible partial transposition* is an operation defined on rank matrices and on the group S_n. Roughly speaking, it consists in transposition of a part of the pillar entries, such that linked pillar entries transpose (or not) simultaneously. More precisely, we have the following:

Definition 3.4. Two rank matrices, $r(w)$ and $r(w')$, are *admissibly partially transpose* to each other if there exists a set $\mathcal{L}$ which is a union of equivalence classes of linked pillar entries for $r(w)$, such that the set of pillar entries of $r(w')$ is as follows

(5)
$$\begin{cases} r'_{ij} & = & r_{ij} & \text{whenever} \quad r_{ij} \notin \mathcal{L} \\ r'_{ji} & = & r_{ij} & \text{whenever} \quad r_{ij} \in \mathcal{L}. \end{cases}$$

Example 3.5. The permutation $w = 11\,2\,9\,8\,6\,4\,5\,12\,3\,7\,10\,1 \in S_{12}$ corresponding to the rank matrix

$$
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 4 \\
0 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 4 & 4 & 5 \\
0 & 1 & 1 & \boxed{2} & 2 & 3 & 3 & 4 & 5 & 5 & 5 & 6 \\
0 & 1 & 1 & 2 & 3 & \boxed{4} & 4 & 5 & \boxed{6} & 6 & \boxed{7} \\
0 & 1 & 1 & 2 & 3 & 4 & 4 & 5 & 6 & 6 & 7 \\
0 & 1 & 2 & 3 & 4 & \boxed{5} & 5 & 6 & 7 & 7 & 8 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \boxed{8} & 8 & 9 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
$$

is admissibly partially transpose to the permutation given in Example 3.2. Indeed, the set of pillar entries is the same except for the last equivalence class for which the positions of the pillar entries are all in the transpose positions.

3.4. **Statement of the main theorem.** In this section we formulate a sufficient condition for the tangent cones of two Schubert varieties to coincide. Furthermore, it turns out that every partial transposition of the pillar entries in the associated rank matrices defines an operation on the group S_n.

Our main result is the following

Theorem 3.6. *(i) Given a permutation $w \in S_n$ and the corresponding rank matrix $r(w)$, for every admissible partial transposition, $r(w)'$, of $r(w)$ there exists a permutation $w' \in S_n$ such that $r(w)' = r(w')$.*

(ii) If w and w' are admissibly partially transpose to each other, then corresponding Schubert varieties have same tangent cones: $\mathcal{T}_w = \mathcal{T}_{w'}$.

We will prove this theorem in Sections 4.5 and 5.7.

Example 3.7. The Coxeter elements of S_4, see Example 2.4, have the same two dissociated pillar entries, 1 and 2, and their positions in the rank matrices differ by transpositions. Therefore, the Schubert varieties corresponding to these elements have the same tangent cone. This statement can be generalized, see below

Corollary 3.8. *Schubert varieties corresponding to the Coxeter elements have the same tangent cone.*

Proof. The pillar entries of Coxeter elements are dissociated and differ by partial transpositions; see Example 2.9. □

Note also that the Schubert varieties corresponding to the Coxeter elements are smooth. Therefore, Corollary 3.8 can also be deduced from the theorem of Lakshmibai, see [12] that describes the Zariski tangent space.

3.5. **Other admissible transpositions.** Theorem 3.6 provides large classes of Schubert varieties with identical tangent cones. However, these classes can be yet larger. In fact, there are other cases of partial transposition of pillar entries than those considered above.

Example 3.9. The permutations $w = 6745321$ and $w' = 6753421$ in S_7 have the following rank matrices:

0	0	0	0	0	0	0	0
0	0	0	0	0	0	(1)	1
0	0	0	0	0	0	1	2
0	0	0	0	(1)	1	2	3
0	0	0	0	1	2	3	4
0	0	0	1	2	3	4	5
0	0	1	2	3	4	5	6
0	1	2	3	4	5	6	7

0	0	0	0	0	0	0	0
0	0	0	0	0	0	(1)	1
0	0	0	0	0	0	1	2
0	0	0	0	0	1	2	3
0	0	0	(1)	1	2	3	4
0	0	0	1	2	3	4	5
0	0	1	2	3	4	5	6
0	1	2	3	4	5	6	7

The rank matrix $r(w)$ has two pillar entries: r_{16} and r_{34}, the interval $[3,4]$ is entirely contained in the interval $[1,6]$. Therefore, these pillar entries of w are related in view of Definition 3.1. However, it is easy to check that $\mathcal{T}_w = \mathcal{T}_{w'}$, in other words, the partial transposition relating w and w' should also be considered as admissible.

This example is not covered by Theorem 3.6 and shows its limits. For instance, it shows that the converse statement to Part (ii) of the theorem is false. Existence of such partial transpositions of pillar entries constitutes the main difficulty in solving the initial classification problem.

4. Combinatorial aspects of rank matrices and pillar entries

In this section we describe the main properties of pillar entries of rank matrices and develop the technique necessary from the proof of our main result.

Recall that the set of pillar entries of a rank matrix $r(w)$ determines the permutation w (see Proposition 2.7). We present two algorithms: that of reconstruction of w from the pillar entries of $r(w)$, and that of calculating of the permutation of w' obtained by some partial transpositions of pillar entries of $r(w)$. This allows us to prove Part (i) of Theorem 3.6.

We also give an explicit formula for the (co)dimension of the Schubert cell C_w in terms of the pillar entries of the rank matrix $r(w)$. This result can be useful for the further study of combinatorics of rank matrices.
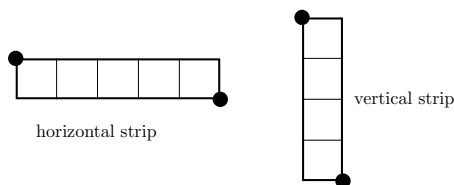
4.1. **Rank matrix and its pillar entries from the permutation diagram.** The rank matrix $r(w)$ is determined by the diagram of the corresponding permutation w.

Proposition 4.1. *One has the following formula:*

(6) $$r_{ij}(w) = \#\{\text{dots in the upper left quadrant from the cell } (i,j)\}.$$

Proof. This readily follows from (1). □

The positions of the pillar entries in $r(w)$ can be determined by local structure of the diagram of w. Consider horizontal strips of height 1 and a vertical strips of width 1 in the diagram, such that the upper left and the lower right corners are marked dots of the permutation:

Proposition 4.2. *Every pillar is located at the intersection of a horizontal strip of height 1 and a vertical strip of width 1.*

Proof. This is a direct consequence of (3). □

Example 4.3. Rank matrix and its pillar entries of $w = 953471682$ in S_9 is as follows.
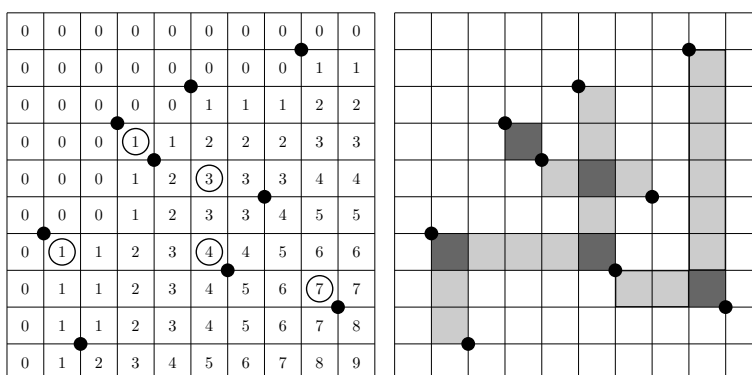
It will be useful in the sequel to have the following observation.

Proposition 4.4. *Every horizontal strip of height 1 necessarily intersects with a vertical strip of width 1, and vice-versa.*

Proof. This statement can be easily proved by induction in the length (height) of the strips. □

4.2. **Reconstructing w from the pillar entries of $r(w)$.** In this section we present an algorithm of construction of the diagram of w from the set of pillar entries.

Let us introduce some useful notation. First we enumerate the pillar entries (from 1 to N) in the "reading order" of their appearance in the diagram of w, i.e., from left to right in each row, and counting the rows from top to bottom. We then set:

$$
\begin{aligned}
(p_i, q_i) \quad &= \quad \text{the position of the } i\text{-th pillar;} \\
K_i := r_{p_i q_i}(w) \quad &= \quad \text{the value of the } i\text{-th pillar} \\
& \qquad \text{i.e. the number of dots in the region at the North-West of } (p_i, q_i); \\
k_i \quad &= \quad \text{the number of dots in the part of the North-West region} \\
& \qquad \text{of the } i\text{-th pillar that does not intersect} \\
& \qquad \text{the North-West regions of the preceding pillar entries.}
\end{aligned}
$$

The diagram of w is constructed in $N + 1$ steps (N is the number of pillar entries).

- The i-th step, for $i \le N$, consists in marking k_i dots from right-to-left and top-to-bottom:
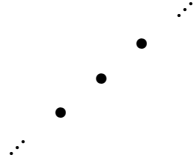
  in the North-West region of the i-th pillar. The dots are placed at the intersections of the columns and rows which are closest to the i-th pillar, that have no marked dot yet, and do not intersect with the North-West regions of all the preceding pillar entries.
- The final, $(N + 1)$-th step consists in marking dots in the same order (from right to left, top to bottom) in the remaining free lines.

The above algorithm is the only way to mark dots without creating an extra pillar or changing the values of the pillar entries. It has some similarities with the algorithm of [7] reconstructing w from the Fulton essential set.

Example 4.5. Figure 3 below illustrates our algorithm for $w = 853471692 \in S_9$. At each step we color the North-West region at the pillar. The dark grey part of the region intersects with North-West regions at previous pillar entries; the light grey part is the area where the new dots are placed.

Remark 4.6. If one defines the following partial ordering on the set of ordered pillar entries

$$
j \prec i \quad \Longleftrightarrow \quad \text{the } j\text{-th pillar lies in the region at the North-West of the } i\text{-th pillar}
$$
$$
\text{i.e. } j < i \text{ , } p_j \le p_i \text{ and } q_j \le q_i
$$

one can write the following relation between the K_i's and k_i's

$$
K_i = k_i + \sum_{j \prec i} k_j.
$$
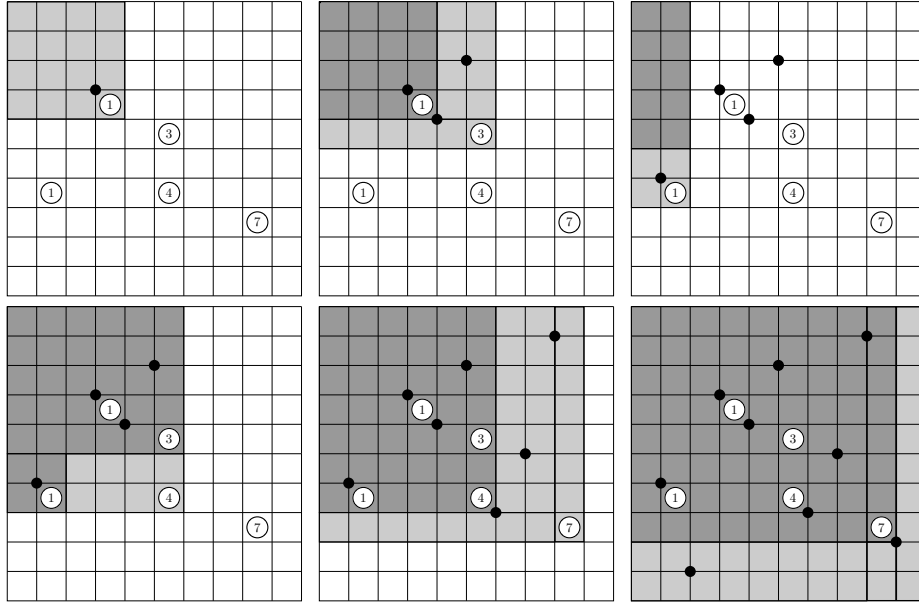
FIGURE 3. Recovering the permutation w from the pillar entries.

4.3. (Co)dimension from the set of pillar entries.

The dimension and codimension of a Schubert cell C_w (or a Schubert variety $\mathcal{X}_w$) can be computed directly form the set of pillar entries of the corresponding rank matrix $r(w)$.

The number

$$\operatorname{codim}(C_w) = \ell(ww_0) = \#\{i < j : w(i) < w(j)\}$$

can be obtained in the diagram of w counting the intersections of the horizontal segments and the vertical segments of the grid that are at the right and above each dots, respectively:

(7) $$\operatorname{codim}(C_w) = \#\{\text{crosses in the diagram of } w\},$$

see Figure 4.

The following formula gives the codimension of a Schubert cell from the data of its pillar entries.

Theorem 4.7. *Using the notation of Section 4.2 one computes*

(8) $$\operatorname{codim}(C_w) = \sum_{i=1}^{N} k_i \left(K_i + n - p_i - q_i\right).$$

Proof. This formula is obtained using the reconstruction algorithm of w from the set of pillar entries (see Section 4.2) and (7). For each dot in the diagram of w, we count the crosses on the horizontal segment at its right. At step i of the construction, the k_i new dots will contribute with the same number of crosses in (7). The reconstruction algorithm of w implies that these crosses can be produced only by the dots that are located at the South-East of the i-th pillar (otherwise it would contradict the fact that one uses the closest available vertical
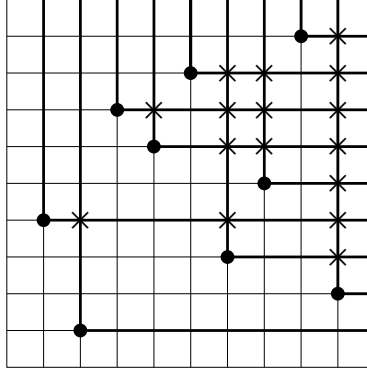
FIGURE 4. $\ell(ww_0)$ from the diagram of w.

lines at the left of the i-th pillar). The number of dots in the South-East area is easy to compute from our data:

$$
\begin{aligned}
\#\{\text{dots at SE}\} &= \#\{\text{dots}\} - \#\{\text{dots at NW}\} - \#\{\text{dots at NE}\} - \#\{\text{dots at SW}\} \\
&= n - K_i - (p_i - K_i) - (q_i - K_i) \\
&= K_i + n - p_i - q_i.
\end{aligned}
$$

Hence the result.   □

4.4. **Truncated permutation.** Given a permutation of $w \in S_n$, we will show the existence of permutations whose pillar entries form subsets in the set of pillar entries of $r(w)$.

The pillar entries of $r(w)$ are decomposed in the disjoint union of equivalence classes of linked pillar entries: $\{r_{ij}(w)\} = \mathcal{L}_1 \sqcup \mathcal{L}_2 \sqcup \ldots \sqcup \mathcal{L}_s$. These classes will be ordered as follows

(9) $u < v \quad \Longrightarrow \quad \min(i,j : r_{ij}(w) \in \mathcal{L}_u) < \min(i,j : r_{ij}(w) \in \mathcal{L}_v).$

The following statement follows from the reconstruction algorithm of Section 4.2.

Proposition 4.8. *For every $t \in \{1, \ldots, s\}$ there exists a unique permutation, denoted $\mathrm{trc}_t(w)$, having $\mathcal{L}_1 \sqcup \mathcal{L}_2 \sqcup \ldots \sqcup \mathcal{L}_t$ as set of pillar entries.*

Proof. This permutation is obtained by stopping the algorithm of reconstruction of w given in Section 4.2 after the step corresponding to the last pillar of the class $\mathcal{L}_t$ and jumping to the final step.   □

Example 4.9. For $w = (12, 2, 9, 7, 6, 4, 10, 5, 3, 11, 1, 8)$ as in Example 3.2, the classes are numbered as follows

$\mathcal{L}_1 = \{r_{22} = 1\}$, $\mathcal{L}_2 = \{r_{64} = 2\}$, $\mathcal{L}_3 = \{r_{67} = 4,\ r_{69} = 5,\ r_{97} = 6,\ r_{9\,10} = 8,\ r_{11\,7} = 7\}$.

One then obtains the truncated permutations

$\mathrm{trc}_1(w) = (12, 2, 3, 11, 10, 9, 8, 7, 6, 5, 4, 1)$, $\mathrm{trc}_2(w) = (12, 2, 11, 10, 9, 4, 8, 7, 6, 5, 3, 1)$.

4.5. **Elementary partial transpositions.** The order (9) suggests a natural series of admissible partial transpositions, such that all the equivalence classes of pillar entries $\mathcal{L}_i$ transpose for i less or equal to some value. In this section we present en explicit algorithm of calculating the resulting permutations. This algorithm is the main ingredient of the proof of Part (i) of Theorem 3.6.

For $t \in \{1, \ldots, s\}$, we define *the elementary partial transposition* $w' = \operatorname{trp}_t(w)$, as the permutation having the following set of pillar entries:

$$\begin{cases} r_{ji}(w') & = & r_{ij}(w), & \text{if} & r_{ij}(w) \in \mathcal{L}_1 \sqcup \ldots \sqcup \mathcal{L}_t; \\ r_{ij}(w') & = & r_{ij}(w), & \text{if} & r_{ij}(w) \in \mathcal{L}_{t+1} \sqcup \ldots \sqcup \mathcal{L}_s. \end{cases}$$

Note that every partial transposition can be obtained as a sequence of elementary partial transpositions.

Given a permutation $w = w_1 w_2 \ldots w_n \in S_n$, the entry w_k of w are separated into two disjoint groups, $I_1 \sqcup I_2$:

$$\begin{cases} w_k \in I_1, & \text{if} & k \leq \max(j), & \text{for pillars} & r_{ij}(w) \in \mathcal{L}_1 \sqcup \ldots \sqcup \mathcal{L}_t; \\ w_k \in I_2, & \text{if} & k > \min(i), & \text{for pillars} & r_{ij}(w) \in \mathcal{L}_{t+1} \sqcup \ldots \sqcup \mathcal{L}_s. \end{cases}$$

The algorithm of calculation the permutation $w' = w'_1 w'_2 \ldots w'_n$, obtained via the above elementary partial transposition, consists in three steps:

(1) keep $w'_k = w_k \in I_1$ if $w_k \leq k$, and $w'_k = w_k \in I_2$ if $w_k \geq k$;
(2) inverse the entries $w_k \in I_1$, i.e., write k at position w_k;
(3) fill the remaining positions in w' in the decreasing order.

The proof of the above algorithm is straightforward.

Example 4.10. For the Coxeter element $w = 2\,3\,4\,1 \in S_4$, the elementary transposition

0	0	0	0	0
0	0	①	1	1
0	0	1	②	2
0	0	1	2	3
0	1	2	3	4

$\longrightarrow$

0	0	0	0	0
0	0	0	1	1
0	①	1	②	2
0	1	1	2	3
0	1	2	3	4

is obtained into three steps:

$$2\,3 \,|\, 4\,1 \quad \to \quad 2\,.\,|\,4\,. \quad \to \quad .\,1\,|\,4\,. \quad \to \quad 3\,1\,|\,4\,2,$$

so that $w' = 3\,1\,4\,2$ is another Coxeter element, already considered in Example 2.4, b).

For every $w \in S_n$, the above algorithm implies the existence of a permutation w' such that the pillar entries of $r(w')$ are obtained by an admissible partial transposition of pillar entries of $r(w)$.

Part (i) of Theorem 3.6 is proved.

5. Proof of the main theorem

In this section, we prove Proposition 2.7 and Part (ii) of Theorem 3.6.

5.1. **A coordinate system.** In the neighborhood of the standard flag F_0, the flag variety $\mathcal{F}$ is identified with the subgroup of unitriangular matrices

(10)
$$
X = \begin{pmatrix}
1 & & & \\
x_{21} & 1 & & \\
\vdots & \ddots & \ddots & \\
x_{n1} & \cdots & x_{n\,n-1} & 1
\end{pmatrix}
$$

This defines a local coordinate system $(x_{21}, \ldots, x_{n\,n-1})$ on $\mathcal{F}$. Given a flag $F \in \mathcal{F}$, every space V_i of F is defined as linear span of the first i columns of the matrix X.

Our next goal is to describe the Schubert cells and Schubert varieties in terms of this coordinate system.

5.2. **Submatrices.** Let M_{ij} be the $(n-j) \times i$ submatrix of X consisting of the last $n-j$ rows and the first i columns.

a) If $i \le j$, then this submatrix is of the form

$$
M_{ij} = \begin{pmatrix}
x_{j+1\,1} & \cdots & x_{j+1\,i} \\
\vdots & & \vdots \\
x_{n1} & \cdots & x_{ni}
\end{pmatrix}.
$$

b) If $i > j$, then the submatrix is as follows

$$
M_{ij} = \begin{pmatrix}
x_{j+1\,1} \cdots x_{j+1\,j} & 1 & & \\
\vdots & & \ddots & \\
x_{i1} & \cdots & x_{i\,i-1} & 1 & \\
\vdots & & & \vdots \\
x_{n1} & \cdots & \cdots & x_{ni}
\end{pmatrix}.
$$

5.3. **Relation to the rank matrices.** The following lemma translates the description of Schubert cells in terms of rank matrices into an algebraic descripiton in the above coordinate system.

Lemma 5.1. *The matrix X represents a flag in the Schubert cell C_w if and only if*

(11)
$$
\mathrm{rank}(M_{ij}) = i - r_{ij}(w),
$$

for all $1 < i \le n$, $1 \le j < n$.

Proof. The space $\mathbb{C}^j$ consists of vectors with zeros at positions $\ge j+1$. One then has

$$
j + \mathrm{rank}(M_{ij}) = \dim(V_i + \mathbb{C}^j) = i + j - r_{ij}(w).
$$

Hence (11). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It follows that the Schubert variety $\mathcal{X}_w$ is represented by the matrix X satisfying the conditions: $\mathrm{rank}(M_{ij}) \le i - r_{ij}(w)$.

5.4. **Systems of equations for $\mathcal{X}_w$ and $\mathcal{T}_w$.** The Schubert variety $\mathcal{X}_w$ is determined, in a neighborhood of the standard flag F_0, by a system of polynomial equations in the variables x_{ij}. The equations are obtained as follows. For each couple of indices i, j, formula (11) leads to a set of equations that expresses the annihilation of the minors of the matrix M_{ij} of size larger than its rank. From Proposition 2.7, it suffices to consider only the equations for the indices i, j corresponding to a pillar entry $r_{ij}(w)$ in the rank matrix of $\mathcal{X}_w$.

The system of equations of the tangent cone $\mathcal{T}_w$ of $\mathcal{X}_w$ is obtained, roughly speaking, as the homogeneous lower degree parts of the equations of $\mathcal{X}_w$. More precisely, the equations of $\mathcal{X}_w$ can be written in such a way that the homogeneous terms of lower degree are linearly independent. Then the system of $\mathcal{T}_w$ is obtained by removing all of the monomials of higher degree in the equations of $\mathcal{X}_w$.

Example 5.2. The first example of a Schubert variety with singularity at the origin correspond to the permutation $w = 4231 \in S_4$ (see [2, 13]). Written in our local coordinates:

$$
\begin{pmatrix}
1 & & & \\
x & 1 & & \\
t & y & 1 & \\
v & u & z & 1
\end{pmatrix}
$$

the equation of the corresponding tangent cone $\mathcal{T}_w$ (the same as the equation of $\mathcal{X}_w$) is: $tu - yv = 0$. Indeed, the rank matrix of w is as follows:

0	0	0	0	0
0	0	0	0	1
0	0	①	1	2
0	0	1	2	3
0	1	2	3	4

so that the Schubert cell C_w is determined by the condition $\dim(V_2 \cap \mathbb{C}^2) = 1$, that translates in coordinates as the condition that a certain linear combination of two first column vectors belong to the subspace $\mathbb{C}^2$, i.e., the matrix M_{22} degenerates.

The tangent cone $\mathcal{T}_w$ is 5-dimensional, whereas the Zariski tangent space is the whole 6-dimensional tangent space $T_{F_0}\mathcal{F}$.

5.5. **The duality.** In the case $i \leq j$, the minors of M_{ij} are homogeneous polynomial expressions. The following observation explains the reason for which two pillar entries transposed to each other, in many situation give the same contribution to the system of equation of the tangent cones.

If $i > j$, then M_{ji} is the complement of the upper right square submatrix in M_{ij} (of size $i - j$) with 1's on the diagonal:

$$
M_{ij} = \left(
\begin{array}{c|c}
 & \begin{matrix} 1 & & \\ \vdots & \ddots & \\ & \cdots & 1 \end{matrix} \\
\hline
M_{ji} &
\end{array}
\right).
$$

The lower degree homogeneous part in the expression of any minors of M_{ij} of size $r \geq i - j$ involving the last $i - j$ columns corresponds precisely to a minor of M_{ji} of size $r - i + j$, and vice versa.

5.6. **Proof of Proposition 2.7.** Let us show that the pillar entries determine the rank matrix. We use the fact that the rank matrix $r(w)$ completely determines the Schubert veriety $\mathcal{X}_w$.

The system of equations of $\mathcal{X}_w$ written in the coordinate system (10) is expressed in terms of vanishing of minors of the matrices M_{ij} that can be calculated according to Lemma 5.1 using every entry r_{ij} of the matrix $r(w)$. However, not all of these equations are relevant. In the following cases:

$$r_{ij} = r_{i-1\,j}, \qquad \text{or} \qquad r_{ij} = r_{i\,j-1},$$

the equations given by r_{ij} follow from the equations given by $r_{i-1\,j}$, or $r_{i\,j-1}$, respectively. In the cases:

$$r_{ij} = r_{i+1\,j} + 1, \qquad \text{or} \qquad r_{ij} = r_{i\,j+1} + 1,$$

the equations given by r_{ij} follow from the equations given by $r_{i+1\,j}$, or $r_{i\,j+1}$, respectively. In each of the above cases, we can remove the equations given by r_{ij} from the system of equations of $\mathcal{X}_w$.

We have proved that the equations of $\mathcal{X}_w$ given by the entry r_{ij} is relevant if and only if r_{ij} is the pillar entry.

5.7. **Proof of Theorem 3.6.** We will need the following lemma[1].

Lemma 5.3. *For every $w \in S_n$, the Schubert varieties $\mathcal{X}_w$ and $\mathcal{X}_{w^{-1}}$ have same tangent cone.*

Proof. The homeomorphism $x \mapsto x^{-1}$ from BwB to $Bw^{-1}B$ induces the isomorphism $f \mapsto -f$ from $\mathcal{T}_w$ to $\mathcal{T}_{w^{-1}}$. $\qquad\square$

We are ready to prove Theorem 3.6, Part (ii). Assume that two permutations, w and w', are admissibly partially transpose to each other. We want to show that the tangent cones of $\mathcal{X}_w$ and $\mathcal{X}_{w'}$ coincide.

We can assume that $w' = \mathrm{trp}_t(w)$ is an elementary partial transposition of w, see Section 4.5 for the definition and the notation. The systems of equations for $\mathcal{X}_w$ and $\mathcal{X}_{w'}$ split in two parts: the equations coming from the pillar entries in the classes $\mathcal{L}_1 \sqcup \ldots \sqcup \mathcal{L}_t$ (these equations are *a priori* different for w and w' since the pillar entries are not in the same positions) and the equations coming from the pillar entries in the other classes, namely in $\mathcal{L}_{t+1} \sqcup \ldots \sqcup \mathcal{L}_s$. The latter equations are identically the same for w and w'.

Consider finally the two subsystems of equations for $\mathcal{X}_w$ and $\mathcal{X}_{w'}$ coming from the pillar entries in the set $\mathcal{L}_1 \sqcup \ldots \sqcup \mathcal{L}_t$. These two subsystems are precisely those describing the Schubert varieties associated to $\mathrm{trc}_t(w)$ and $\mathrm{trc}_t(w')$, respectively. These two varieties have same tangent cones since $\mathrm{trc}_t(w) = \mathrm{trc}_t(w')^{-1}$. After intersecting with the tangent cone of the variety described by the rest of the system, one therefore obtains the same tangent cone for $\mathcal{X}_w$ and $\mathcal{X}_{w'}$.

Theorem 3.6 is proved.

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[1]We are grateful to M. Kashiwara for a simple proof.

6. Enumerative results

Theorems 3.6 gives an efficient method for calculating the number of different tangent cones of Schubert varieties. In this section, we give the result in small dimensions and codimensions. The general result is still out of reach.

Recall that the total number of Schubert varieties is $n!$, the total number of their tangent cones is smaller. It would be interesting to find asymptotic of the number of tangent cones.

6.1. **Low-dimensional cases.** In the case $n = 4$, the comparative number of Schubert varieties and their tangent cones, as a function of their dimension, is given by the following table.

dim	0	1	2	3	4	5	6
Schub	1	3	5	6	5	3	1
TangCones	1	3	3	3	3	2	1

The total number of tangent cones in this case is 16.

For $n = 5$, the table is:

dim	0	1	2	3	4	5	6	7	8	9	10
Schub	1	4	9	15	20	22	20	15	9	4	1
TangCones	1	4	6	7	9	9	10	8	6	2	1

The total number of tangent cones is 63.

For $n = 6$, the distribution of the tangent cones is as follows:

dim	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
TangCones	1	5	10	14	20	25	31	36	40	40	34	24	15	8	3	1

The total number of tangent cones for $n = 6$ is 343.

For $n = 7$ and 8, the total numbers of tangent cones are: 1821 and 13041, respectively[2]. Note that the sequence $16, 63, 343, 1821, 13041, \ldots$ does not appear in Sloane's online Encyclopedia of Integer Sequences.

6.2. **Tangent cones of codimension** 2. Let us also consider the case of small codimension.

The tangent cone of the Schubert variety $\mathcal{X}_{w_0}$ corresponding to longest element $w_0 \in S_n$, is the only one tangent cone of dimension $\frac{n(n-1)}{2}$.

Next, in the case of dimension $\frac{n(n-1)}{2} - 1$ (*i.e.*, of codimension 1), there are $n-1$ Schubert varieties that have $\left[\frac{n}{2}\right]$ tangent cones. Indeed, the elements $\mathcal{X}_w$ and $\mathcal{X}_{w^{-1}}$ have the same tangent cone.

There are $\frac{(n+2)(n-1)}{2}$ Schubert varieties of codimension 2. The number of their tangent cones depend on the parity of n, as given by the following statement.

Proposition 6.1. *The number of tangent cones of codimension 2 is:*

$$2 + \frac{(n-3)(n+11)}{8}, \qquad and \qquad 3 + \frac{(n-4)(n+14)}{8},$$

for odd n, and for even n, respectively.

Proof. A straightforward calculation. □

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[2]These numbers are obtained using computer programs.

Appendix

A1 Comparison of pillar entries to essential entries. Relationship between pillar entries and Fulton's essential entries is not yet completely understood. Below are a series examples that (partly) clarify the situation. Recall that essential entries are boxed (while pillar entries are encercled as above).

Let us consider examples that emphasize the difference between the notions of essential and pillar entries. The most interesting case is that of the Coxeter elements.

Example 6.2. a) The rank matrix of the element $w_0 = 4\,3\,2\,1$ in S_4 has three essential entries

0	0	0	0	0
0	0	0	$\boxed{0}$	1
0	0	$\boxed{0}$	1	2
0	$\boxed{0}$	1	2	3
0	1	2	3	4

and no pillar entries. It can be deduced from formula (1), that, for an arbitrary n, the only rank matrix without pillar entries is the matrix $r(w_0)$ of the longest element $w_0 \in S_n$. This matrix has $n - 2$ essential entries along the antidiagonal.

b) For each of the elements $w_1 = 2\,1\,4\,3$ and $w_2 = 4\,2\,3\,1$ of S_4, we have two essential entries and one pillar:

0	0	0	0	0
0	$\boxed{0}$	1	1	1
0	1	$②$	2	2
0	1	2	$\boxed{2}$	3
0	1	2	3	4

and

0	0	0	0	0
0	0	0	$\boxed{0}$	1
0	0	$①$	1	2
0	$\boxed{0}$	1	2	3
0	1	2	3	4

Note that the position of the pillar entry in the above matrices is the same, while those of the essential entries are different.

c) For the Coxeter elements of S_4, we have:

0	0	0	0	0
0	0	$①$	1	1
0	0	1	$②$	2
0	$\boxed{0}$	1	2	3
0	1	2	3	4

0	0	0	0	0
0	0	$①$	1	1
0	$\boxed{0}$	1	$\boxed{1}$	2
0	1	$②$	2	3
0	1	2	3	4

0	0	0	0	0
0	0	$\boxed{0}$	1	1
0	$①$	1	$②$	2
0	1	$\boxed{1}$	2	3
0	1	2	3	4

0	0	0	0	0
0	0	0	$\boxed{0}$	1
0	$①$	1	1	2
0	1	$②$	2	3
0	1	2	3	4

A2 Rothe diagrams and opposite Rothe diagrams. The Rothe diagram [16] of a permutation $w \in S_n$ is an $n \times n$ square table obtained according to the following rule. Dot the cell (i, j) whenever $w(i) = j$, shade all the cells of the row at the right of the dotted cell and all the cells of the column below the dotted cell (including the dotted cell). Note that the length $\ell(w)$ is equal to the number of white cells in the Rothe diagram.

It was noticed in [8], that the white cells having a South and East frontier with the shaded region give the positions of the essential entries in the corresponding rank matrix. The value of an essential entry is equal to the number of dots in the upper left quadrant of the Rothe

diagram with the origin at the corresponding cell. Let us explain a similar rule to obtain positions of pillar entries.
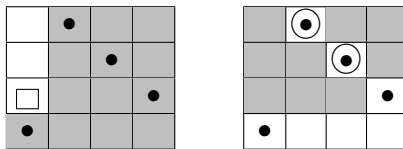

TABLE 1. The Rothe diagram (left) and the opposite Rothe diagram (right) of the Coxeter permutation $2\,3\,4\,1$. The Rothe diagram gives the unique essential entry in the rank matrix: $r_{3,1} = 0$, whereas the opposite diagram gives two pillar entries: $r_{1,2} = 1$ and $r_{2,3} = 2$.

Consider the *opposite Rothe diagram* obtained with the following rule. Shade all the cells of the row strictlty at the left of the dotted cell and all the cells of the column strictly above the dotted cell (the dotted cell is not shaded). Note that the number of white undotted cells in the opposite Rothe diagram is equal to $\ell(w)$.

It follows directly from Definition 2.6, that the white cells having a South and East frontier with the shaded region in the opposite Rothe diagram give the positions of the pillar entries in the corresponding rank matrix. The value of a pillar entry is equal to the number of dots in the upper left quadrant of the diagram.

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