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Étude des q-analogues de nombres réels : aspects combinatoires et géométriques, extensions, généralisation en dimension supérieure

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À mon grand-père, qui m'a montré la voie.

Nous sommes les pensées arborescentes
qui fleurissent sur les jardins cérébraux

P'oasis, Corps et biens, Robert Desnos

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Résumé

Un analogue quantique est une déformation d'un objet mathématique réalisée par l'ajout d'un paramètre formel, traditionnellement noté q. La spécialisation $q = 1$ doit redonner l'objet de départ, qui se voit ainsi plongé au sein d'une famille d'objets similaires. De nombreux objets ont des q-analogues naturels, et en particulier les entiers, dont les plus célèbres déformations remontent aux travaux d'Euler et Gauss sur les q-séries. En 2020, Morier-Genoud et Ovsienko ont introduit une déformation des nombres réels qui étend la notion de q-entier. Construits par des méthodes combinatoires, ces q-réels possèdent des propriétés inattendues en lien avec de nombreux domaines. L'objectif de cette thèse est double : poursuivre l'exploration de la théorie des q-nombres, et proposer des généralisations de cette théorie pour augmenter sa portée. Nous portons ici une attention particulière aux symétries modulaires régissant les q-réels, qui leurs confèrent une cohérence interne. Nous construisons algébriquement deux extensions successives de ces symétries, et nous les utilisons ensuite pour démontrer des résultats combinatoires sur les rationnels quantiques. Nous illustrons la richesse de la combinatoire des q-nombres en étudiant des analogues naturels des nombres de Markov, donnés par les traces de certaines matrices déformées. Nous changeons ensuite de point de vue en explorant l'effet de transformations géométriques sur les q-nombres, interprétés comme des disques dans le plan. Enfin, nous mettons à profit la théorie des représentations de tresses pour introduire une généralisation en dimension 2 des analogues quantiques de rationnels.

Mots clés : Analogues quantiques, q-déformation, nombres de Markov, pavage de Farey, groupe modulaire, groupe de tresses, représentation de Burau, graphe en serpent.

Abstract

A quantum analogue is a deformation of a mathematical object by the addition of a formal parameter, tradionnally denoted by q. The specialization $q = 1$ must give back the object, which is thus embedded in a family of similar objects. Many objects have natural q-analogues, in particular integers. The most famous deformation of integers goes back to Euler and Gauss who used it to study q-series. In 2020, Morier-Genoud and Ovsienko introduced a deformation of real numbers extending the notion of q-integers. Defined by combinatorial methods, these q-reals satisfy unexpected properties related to several areas of mathematics. The goal of this thesis is twofold : to continue the exploration of the theory of q-real numbers, and to define generalizations of this theory. We pay here particular attention to the modular symmetries underlying q-real numbers. We build algebraically two successive extensions of these symmetries, and we use them to prove combinatorial results on q-rational numbers. We illustrate the rich combinatorics of q-rationals by studying natural analogues of Markov numbers, given by the traces of some q-deformed matrices. We then shift our point of view and explore the effect of some geometric transformations on q-real numbers, that we interpret as disks in the plane. Finally, we take advantage of the theory of braid groups representations to introduce a dimension 2 generalization of quantum rational numbers.

Key words : Quantum analogues, q-deformation, Markov numbers, Farey tesselation, modular group, braid group, Burau representation, snake graph.

Table des matières

Contents

List of Figures

Introduction

No one knows the reason for all this,
but it is probably quantum.

Pyramids, Terry Pratchett

State of the art: a short survey on q-real numbers

Quantum deformation is a mathematical concept appearing in many different areas such as representation theory, knot theory, quantum calculus... The idea is to enrich an object by embedding it into a one parameter family, the parameter being traditionally denoted by q. The specialization $q \to 1$ must give back the object that is being deformed. For a deformation to be interesting, it should be relevant with respect to the mathematical context to which the object belongs. In combinatorics, enumerative sequences of integers such as factorials or binomial coefficients possess a natural q-deformed version, giving a more precise enumeration of what these sequences are counting. They are built using Euler and Gauss q-integers, a very well-known family of polynomials considered as the quantum version of nonnegative integers.

$$[n]_q := \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1},\ n \in \mathbb{N}. \tag{1}$$

This deformation is extended to negative integers by $[-n]_q = \frac{q^{-n}-1}{q-1} = -q^{-1}[n]_{q^{-1}}$. The combinatorics of q-integers and related deformations is an active research area, and almost all classical sequences of integer have a q-analogue, such as q-binomial coefficients or q-Catalan numbers for example.

In the 2020's, Sophie Morier-Genoud and Valentin Ovsienko defined an extension of q-integers to real numbers. It might be surprising that this has not been done before, but that is mostly because naive ideas fail, and it was not obvious how to proceed. Let us explain briefly in what sense. On one hand, replacing the integer n by a rational number x in the formula of q-integers (1) leads to fractional powers of q, which does not behave nicely. On the other hand, defining $\left[\frac{a}{b}\right]_q = \frac{[a]_q}{[b]_q}$ leads to a notion of q-deformed rational line that does not preserves the natural order of rational numbers. Indeed, for two rational numbers $\frac{a}{b} < \frac{c}{d}$, the quantity $[c]_q[b]_q - [a]_q[d]_q$ does not have positive coefficients in general, which breaks the order.

The definition of q-rational numbers of Morier-Genoud and Ovsienko avoids these pitfalls, and turns out to be relevant in a wide range of fields, from homological algebra to knot theory, from Diophantine approximation to cluster algebra. Before focusing on

the work presented in this thesis, we give an overview of the literature on q-real numbers, which has been growing very fast since the original article of 2020. This is also the occasion to introduce the main objects and questions we are going to work with in this document.

Construction of q-real numbers

The most elementary definition of q-real numbers uses continued fraction expansions. If $x \in \mathbb{R}$, there is a unique sequence of integers $(c_1, c_2, c_3, \cdots)$, with $c_i \geq 2$ for all $i \geq 2$ (and without k such that $c_i = 2$ for all $i \geq k$) such that

$$x = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cfrac{1}{\ddots}}}.$$

This is called the negative, or the Hirzebruch-Jung continued fraction of x. When x is a rational number, the sequence $(c_i)_i$ is finite so the formula above is well-defined and we denote $x = [\![c_1, c_2, \cdots, c_k]\!]$. When x is an irrational number, the formula above is a notation meaning that the sequence of convergents $(x_n = [\![c_1, c_2, \cdots, c_n]\!])_n$ converges toward x.

The q-analogue of x is computed via a deformation of its continued fraction expansion [MGO20].

$$[x]_q = [c_1]_q - \cfrac{q^{c_1-1}}{[c_2]_q - \cfrac{q^{c_2-1}}{[c_3]_q - \cfrac{q^{c_3-1}}{\ddots}}}. \tag{2}$$

Hence, if x is a rational number, its q-analogue is a rational fraction in q, evaluating at x when $q = 1$. If x is an irrational number, the formula above is a notation meaning that the sequence of q-convergents $([x_n]_q = [[\![c_1, c_2, \cdots, c_n]\!]]_q)_n$ converges in the q-adic topology toward a formal Laurent power series [MGO22]. Note that in the case of irrational numbers, it does not make sense anymore to evaluate q without further study.

x	$\xrightarrow{q}$	$[x]_q$
$\mathbb{N}$		$\mathbb{Z}_{\geq 0}[q]$
$\mathbb{Z}$		$\mathbb{Z}[q, q^{-1}]$
$\mathbb{Q}_{\geq 0}$		$\mathbb{Z}_{\geq 0}(q)$
$\mathbb{Q}$		$\mathbb{Z}(q)$
$\mathbb{R}_{\geq 0}$		$\mathbb{Z}[[q]]$
$\mathbb{R}$		$\mathbb{Z}[[q]][q^{-1}]$

Figure 1: Spaces of q-analogues of numbers

This deformation of real numbers was not chosen arbitrarily. The main interest of this definition is that it endows q-real numbers with a q-deformed modular structure, as already

suggested in [MGO20]. Leclere and Morier-Genoud then proved that there is a q-deformed action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on the q-real line [LMG21], generated by

$$T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}.$$

With these two q-deformed generators of the modular group, one can check that any matrix $M \in \mathrm{PSL}_2(\mathbb{Z})$ can be replaced by a q-version $M_q \in \mathrm{PGL}_2(\mathbb{Z}[q, q^{-1}])$, and this quantization commutes with the deformation of real numbers:

$$\forall x \in \mathbb{R},\ [M \cdot x]_q = M_q \cdot [x]_q. \tag{3}$$

Using this equivariance property, Leclere and Morier-Genoud were able to show that for any choice of coefficients c_i in the continued fraction expansion of x, the quantization formula (2) gives the same result $[x]_q$.
Surprisingly, another deformation of rational numbers happens to satisfy the same equivariance property. This is due to the fact that the operator T_q has two different fixed points in $\mathbb{P}^1(\mathbb{Z}(q))$, the infinite fraction $\frac{1}{0}$ and another one, $\frac{1}{1-q}$. The twin version of q-rationals was studied by Bapat, Becker and Licata [BBL23] and is called the left version. Thus, a rational number x has two natural q-analogues, the so-called right version, denoted by $[x]_q^\sharp$, computed by (2) and the left version, denoted by $[x]_q^\flat$, given by a slight modification in the end of the q-continued fraction formula:

$$[x]_q^\flat = [c_1]_q - \cfrac{q^{c_1-1}}{[c_2]_q - \cfrac{q^{c_2-1}}{\ddots \cfrac{}{[c_{k-1}]_q - \cfrac{q^{c_{k-1}-1}}{[c_k]_q^\flat}}}}, \tag{4}$$

where $[n]_q^\flat = \frac{q^{n+1}-q^n+q^{n-1}-1}{q-1} = 1 + q + \cdots + q^{n-2} + q^n$ for $n \in \mathbb{N}$.

Asymptotically, the left and the right versions coincide, so that there is only one version of q-irrational number, limit of both left and right q-deformations.

Combinatorics of q-rationals

Thanks to the modular symmetries (3), the q-analogues of rational numbers have a very nice combinatorial behaviour. We enumerate below different models related to q-rationals. This list may not be exhaustive but is aimed to give a wide overview of the combinatorics behind q-rationals. Some of these models will be explained in more detail in the main body of this document.

- Farey graph: it is an infinite graph whose vertices are nonnegative rational numbers and edges relate pairs of rationals $(\frac{a}{b}, \frac{c}{d})$ such that $|ad - bc| = 1$. The q-rational numbers can be computed using a weighted Farey graph, described for the right version in [MGO20]. It is a powerful tool, used for example to prove the property of total positivity : if two rational numbers are ordered $\frac{a}{b} < \frac{c}{d}$, then their right q-deformations $\frac{A}{B}$ and $\frac{C}{D}$ satisfy $BC - AD \in \mathbb{Z}_{>0}[q]$. It also provides a combinatorial interpretation of q-rationals in terms of paths in an oriented version of the Farey

graph [LMG23]. The left version of this model was discussed in [FQ25, Ren24]. In Section 1.4.3 and Section 3.2, we will generalize the weighted Farey operations and prove other total positivity results.

- Fence posets or linear quivers: via a correspondence between paths in the Farey graph and ordered ideals of fence posets, one can interpret q-rationals as generating functions of the latter. This was stated (in the language of closures in type A quivers) in [MGO20], and this point of view became the main tool to prove the unimodality conjecture : the numerators and denominators of right q-rational numbers are unimodal. The first steps toward this conjecture were made by McConville, Sagan and Smyth [MSS21], and then a full proof was given by Kantarcı Oğuz and Ravichandran, based on the creation of circular fence posets [KOR23]. They prove at the same time that rank generating functions of circular fence posets are symmetric, which corresponds to the palindromicity property of traces of q-matrices [LMG23]. While the proof of [KOR23] was inductive, Elizalde and Sagan found a bijective proof of symmetry of rank generating functions of circular fence posets in [ES23]. We will study a left version of these circular fence posets in Section 2.1. The unimodality of generating functions of fence posets can be understood in the more general setting of F-polynomials in cluster algebras from surfaces, as shown recently by Kang, Lee and Lim [KLL26].

- Snake graphs: as evidenced by the work of Çanakçı, Schiffler [CS20], the combinatorics of continued fractions can be translated into the model of snake graphs. This point of view was taken by Ovenhouse in [Ove23] to give a new interpretation of q-rational numbers in terms of the number of points in some Schubert cells over finite fields. To our knowledge, it is the only connection (at least for now) between q-rational numbers and geometry over finite fields. In an other context, Burcroff, Ovenhouse, Schiffler and Zhang proposed a higher-dimensional generalization of q-rational numbers [BOSZ24] by looking at m-dimers in snake graphs, taking advantage of the duality between m-dimers in a snake graph and m-lattice paths in an other snake graph (the dual one). They proved higher versions of total positivity and convergence properties of q-rationals. Finally, very recently a new model for right q-rational numbers was introduced by Ovsienko, based on weighted counting of dimers in snake graphs [Ovs25]. This model of weighted dimers was originally developed to study q-Markov numbers in [EJMGO26], and we will present this work in Chapter 2.

- Frieze patterns: soon after the appearance of q-rational numbers, Morier-Genoud and Ovsienko applied their new theory to Conway-Coxeter friezes. They defined a notion of q-frieze, and show analogue results of the classical combinatorics of (type A) friezes in the quantum setting [MGO21]. This notion of q-friezes was investigated in details and slightly extended recently by Inuma, Motomiya and Kogiso [IMK26].

- Hyperbinary partitions: McConville, Propp and Sagan rose up a new isomorphism between the lattice of hyperbinary partitions of an integer n and the lattice of ordered ideals of some associated fence poset [MPS26]. Their work hence connects the combinatorics of hyperbinary partitions to the theory of q-rational numbers, and they deduce from this connection new computational methods to handle q-rationals.

- Ostrowski's numeration system: this last model was pointed out by Aval and Labbé, considering a modified version of a numeration system studied by Ostrowski, based on a notion of admissible sequences attached to a finite continued fraction expansion [AL25]. They show that q-rationals can be interpretated as generating functions of such admissible sequences. This leads them to upgrade the models of fence posets and snake graphs associated to q-rational numbers so that they can describe all positive rational numbers and not only the ones greater than 1 as before. In this document, we use their upgraded models, see for instance Section 1.4.

Analytical aspects of q-irrational numbers

With the definition of Morier-Genoud and Ovsienko [MGO22], the q-analogue of an irrational number is a formal power series in q. It is natural to ask about the radius of convergence of such a series and whether a closed form can be computed or not. In this perspective, the variable q is not an indeterminate anymore but a real or complex parameter. First attempts in these directions were made by Morier-Genoud and Ovsienko [MGO22] in the case of quadratic irrational numbers, and this case was entirely solved soon after by Leclere and Morier-Genoud [LMG21]. If $x = \frac{a\pm\sqrt{d}}{b}$ is a quadratic irrational number, then its q - deformation is solution of a polynomial equation of degree 2, and thus can be written in the form

$$\left[\frac{a \pm \sqrt{d}}{b}\right]_q = \frac{A \pm \sqrt{D}}{B}, \tag{5}$$

with A, B, D three Laurent polynomials in q. The key tool of the proof is the equivariant q-action of the modular group on q-real numbers. Studying traces of the q-matrices defined by this action, they could also prove that the polynomial D is palindromic.

For a quadratic irrational number x, the radius of convergence of $[x]_q$ is then directly related to the roots of polynomials D and B in (5). The question of localisation of these roots was investigated by Leclere, Morier-Genoud, Ovsienko and Veselov in the particular cases of the golden ratio φ and the silver ratio $\sqrt{2}$, computing explicitely their radius of convergence [LMGOV24]. They conjecture that among all real numbers, $[\varphi]_q$ has the lowest radius of convergence, which is $R_\star = \frac{3-\sqrt{5}}{2} \simeq 0.38$. They attack the conjecture by considering rational numbers, proving that under some hypothesis on $x \in \mathbb{Q}$, the radius of $[x]_q^\sharp$ is indeed greater then $R_\star$. They also get a weaker version, valid for all rational numbers: the radius of convergence of $[x]_q^\sharp$ is greater than $3 - 2\sqrt{2} \simeq 0.17$, for any $x \in \mathbb{Q}$. Their proof relies on showing that the roots of the numerator and denominator of $[x]_q^\sharp$ are contained in the open annulus $\{3 - 2\sqrt{2} < |q| < 3 + 2\sqrt{2}\}$. This theorem was extended to quadratic irrational numbers by Leclere in her thesis [Lec24]. Let us mention that this fact was used later in [MGOV24] to study specializations of the Burau representation of B_3.

A few months later the publication of [LMGOV24], Ren proved the conjecture for metallic numbers [Ren22], a family of quadratic irrational numbers whose regular continued fraction expansions are $[n, n, n, \cdots]$ for $n \in \mathbb{Z}_{\geq 1}$. The first metallic number is the golden ratio, the second one is the silver ratio $\sqrt{2}$. In addition to the result for all metallic numbers, Ren also proved that the conjecture holds for all the convergents $[n, n \cdots, n]$.

In 2024, the conjecture was proven for rational numbers by Elzenaar, Gong, Martin and Schillewart, using geometrical methods [EGMS25]. About the radius of q-rational numbers, we might also mention that Evans, Veselov and Winn [EVW24] investigated the rational numbers with maximal radius of convergence (that is 1). They call them Kronecker fractions and prove that they must have palindromic continued fraction expansions, which enables them to classify the ones with prime denominator.

Finally, in 2026, Etingof found an open region $\mathcal{D}$, containing the disk of radius $3-2\sqrt{2}$, in which he could prove that any q-real number converges to a holomorphic non-vanishing function [Eti25, Theorem 1.1]. He showed that this also happens inside the disk of radius $2-\sqrt{3} \simeq 0.27$. The two theorems are complementary, see Figure 2. Besides, he deduced that for any real number x, the function $q \mapsto [x]_q$ is well-defined and real analytic on the open interval $(-R_\star, 1)$, strenghtening the conjecture of [LMGOV24]. As a byproduct, this gives sense to the specialization $\lim_{q\mapsto 1}[x]_q = x$, for irrational x.

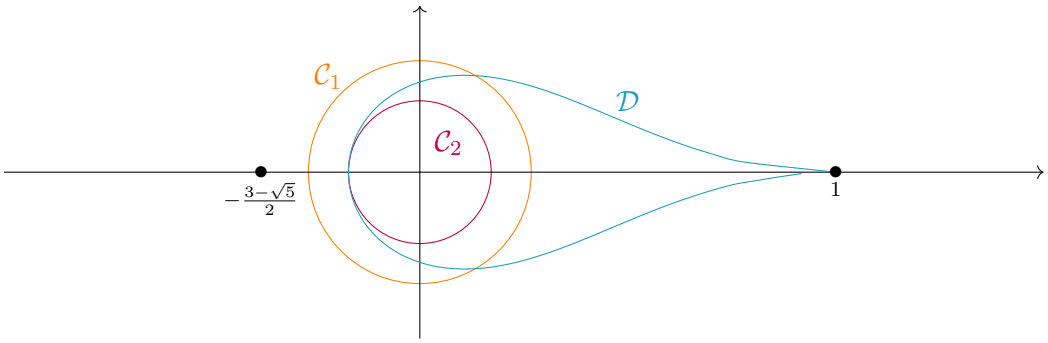


Figure 2: The open region $\mathcal{D}$ (in blue) in which any q-real number $[x]_q$ converges. $\mathcal{C}_1$ (in orange) is the circle of radius $2-\sqrt{3}$, in which there is also convergence of any q-real. $\mathcal{C}_2 \subset \mathcal{D}$ (in purple) is the circle of radius $3-2\sqrt{2}$.

Apart from the radius of convergence of $[x]_q$, there were other interesting works on the analysis of q-real numbers. In [Las25], Lasker studies the function $q \in \mathbb{R} \mapsto [x]_q^\sharp$, around $q=1$, for $x \in \mathbb{Q}$. By design, $[x]_q^\sharp(q=1) = x$, but Lasker computes explicit formulas for the first and the second derivatives of this function, at $q=1$, in relation to the Thomae's function. The proofs follow ingenious inductions in the weighted Farey graph. The function $q \mapsto [x]_q$ was also studied by Etingof, for any real number x and with the variable q contained in a region of convergence of the series. In particular, he proved that the derivative (with respect to q) of $[x]_q$ is nonnegative for $0<q<1$ [Eti25, Proposition 5.3].
On the other side, the function $x \mapsto [x]_q$ has been shown to be strictly increasing on $\mathbb{R}$, for $0<q<1$. For rational numbers, this is a consequence of the total positivity, and for irrational numbers it was formulated by Bapat, Becker and Licata during their study of left q-rational numbers. Their approach implies indeed that for any rational numbers $x<y$, the q-deformations of x and y are well-ordered

$$[x]_q^\flat < [x]_q^\sharp < [y]_q^\flat < [y]_q^\sharp.$$

This explains actually the names of left and right q-rationals. Bapat, Becker and Licata proved that for a fixed $q \in (0,1)$, the function $x \mapsto [x]_q^\sharp$ is right-continuous on $\mathbb{R}$. More

precisely, they show that for any sequence $(x_n)_n$ of rational numbers converging toward $x \in \mathbb{R}$, four cases must be distinguished (this situation was already observed by Morier-Genoud and Ovsienko, see Remark 3.2 in [MGO22]):

1. if $x_n > x$ for all n, and $x \in \mathbb{Q}$, then $[x_n]^\sharp \underset{n\mapsto+\infty}{\longrightarrow} [x]_q^\sharp$;
2. if $x_n < x$ for all n, and $x \in \mathbb{Q}$, then $[x_n]^\sharp \underset{n\mapsto+\infty}{\longrightarrow} [x]_q^\flat$;
3. if $x \in \mathbb{R} \setminus \mathbb{Q}$, then $[x_n]^\sharp \underset{n\mapsto+\infty}{\longrightarrow} [x]_q$;
4. else, the sequence $([x_n]_q^\sharp)_n$ does not converge.

Analogous statements hold for sequences of left q-rational numbers, see Section 1.5.
This result was extended by Etingof to any fixed parameter $q \in \mathcal{D}$. Besides, back with a real parameter $q \in (0,1)$, he proved that the function $x \mapsto [x]_q$ is differentiable with almost everywhere null derivative on $\mathbb{R}$.

On a more number theoretic perspective, the coefficients of the series $[x]_q$ were investigated by Ovsienko, Pedon [OP25] and then Han, Pedon [HP25]. They study the Hankel determinants of the series given by metallic numbers, showing interesting periodicity results and Somos-Gale-Robinson recurrence formulas. Very recently, Pedon established many identities satisfied by coefficients of q-metallic numbers, asymptotic behaviour, logarithmic growth and an intriguing relationship with RNA structures [Ped26].

Besides, another natural question is whether q-algebraic numbers are solutions of some polynomial q-equations. The case of quadratic irrationals, entirely solved with a positive answer in [LMG21], gave hopes but it turned out to be a challenging problem to generalize this in higher degree. Ovsienko and Ustinov worked on the case of some cubic equations in [OU25]. We tackle the degrees 4 and 6 in Section 1.5.

Furthermore, Etingof established a notion of q-complex numbers as a meromorphic continuation of the quantization map on real numbers [Eti25, Section 7].

q-Real numbers in knot theory

One of the origin of q-rational numbers is the connection between cluster algebras and rational knots, that has been made by Lee and Schiffler [LS19]. The same kind of combinatorics was also pointed out by Kogiso and Wakui when they described a bridge between friezes and rational tangles [KW19]. It turns out that this is the same combinatorics underlying the construction of right q-rationals [MGO20]. This was discovered independantly by Sikora, when he studied the Jones Unknot conjecture in the particular case of rational framed tangles [Sik24]. Among various things, he shows that the Kauffman bracket polynomial can be used to compute q-rational numbers (or conversely, q-rational numbers can be used to compute the Kauffman bracket polynomial of rational tangles). Some papers have been outlining the consequences of this relationship since its discovery [Wak25, KMR$^+$25, RY25, BRY26].

On another note, the famous Burau representation realises a link between braid theory and

q-deformed real numbers, as it was first observed by Morier-Genoud, Ovsienko and Veselov [MGOV24]. In this article, they exploit this link to identify which specializations of the Burau representation of B_3 are faithful. The faithfulness of the Burau representations is an important question in knot theory. It has been known for a long time that the Burau representations of B_2 and B_3 are faithful. On the contrary, after a lot of work, Moody [Moo91] and then Long, Paton [LP93] and finally Bigelow [Big99] were able to prove that for $n \geq 5$, the Burau representation of B_n is not faithful. Still remains open the case of B_4, which is related to the Jones Unknot conjecture. During this thesis, we investigated the natural action of B_4 on the rational plane, given by the Burau representation at $t = -1$, aiming to generalize the deformation of rational numbers in dimension 2 [Jou25a]. This work is presented in Chapter 4.

Very recently, a new connection between q-rationals and knot theory has been implemented, building a family of link invariants indexed by $x \in \mathbb{Q}$ [JQ26]. These invariants are not totally new, since they turned out to be specializations of the HOMFLY-PT polynomial. They still provide a meaningful interpolation of the well-known Reshetikhin-Turaev $\mathfrak{sl}_n$-invariants, parametrized by $x = n \in \mathbb{N}^*$.

Arithmetics of q-real numbers

The theory of q-deformed real numbers obviously has intersections with number theory. The first remarkable link is about Markov numbers, a famous family of solutions of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

Several quantization of these numbers exist (for more details, see the introduction of Chapter 2). The q-rational numbers provide a natural substrate to q-deform Markov numbers, which has been investigated by many authors [Kog20, LL22, KO25, LLS24, EJMGO26].

In the same flavour, Mathevet, Morier-Genoud and Ovsienko worked recently on a deformation of Pythagorean triples [MMGO26], introducing a q-deformed Pythagorean equation and constructing a class of solutions satisfying some combinatorial conditions.

Tools from number theory were also applied to q-rational numbers, for example by Uludağ and Yılmaz who studied generalizations of the functions numerator and denominator on rational numbers, reducing to q-rationals in a special case [UY22].

The q-real number $[x]_q$ is a relevant alternative to the expression $\frac{q^x-1}{q-1}$ in the context of q-calculus, as demonstrated by Machacek and Ovenhouse [MO24]. They show many q - identities satisfied by q-real numbers, and in particular they define and study a q-analogue of the Gamma function.

Thanks to a careful study of continued fractions and the fence poset interpretation of q-rational numbers, Kogiso, Miyamoto, Ren, Wakui and Yanagawa proved in [KMR$^+$25] a sufficient condition for denominators of two q-rationals to be equal. They also characterize numerators and denominators of q-rationals that are palindromic.

The q-modular group and its generalizations

Because the construction of q-real numbers of Morier-Genoud and Ovsienko relies on the deformation of the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on the real line $\mathbb{P}^1(\mathbb{R})$, it is very tempting to generalize this process to complex numbers, by extending the group from $\mathrm{PSL}_2(\mathbb{Z})$ to the so-called Picard group $\mathrm{PSL}_2(\mathbb{Z}[i])$. This was investigated by Ovsienko in [Ovs21], where he studied a q-analogue of the imaginary translation operator $z \mapsto z + i$, uniquely determined by compatibility with the q-version of the modular group. Using this, he defined q-deformed Gaussian integers and computed formulas for them, in relation to Chebyshev polynomials. However, in his meromorphic definition of q-complex numbers, Etingof finds other q-analogues of Gaussian integers, only coinciding with the approach of Ovsienko on the elliptic points of the upper complex plane.

The q-deformed action of $\mathrm{PSL}_2(\mathbb{Z})$ on the real line was also taken as a starting point by Alexander Thomas in [Tho24] to define a q-version of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, that is a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ but defined over the ring of polynomials in q. He interprets elements of $\mathfrak{sl}_2(\mathbb{R})$ as differential operators, and by looking at the eigenfunctions of the q-deformed operator $x\partial$, he finds a Möbius transformation called the transition map, bridging the gap between left and right versions of q-rational numbers. In this context, he also defined and studied q-deformations of the Heisenberg and the Witt algebras.

Moreover, the question of the specializations of the q-deformed matrices was already raised in the original paper of Morier-Genoud and Ovsienko [MGO20], where they show that at $q = -1$, the q-deformed modular group collapses to the symmetric group $\mathfrak{S}_3$. In her PhD thesis [Lec24], Leclere worked on specializations of q at roots of unity, and her results were independantly stated later by Byakuno, Ren and Yanagawa [BRY26]. In short, the result is that the q-deformed modular group is finite if and only if q is specialized to a nth root of unity, for $n \in [\![2, 5]\!]$, and in these cases it is reduced to a group of isometries of a regular polyhedron.

A notion of q-transpose, compatible with the q-deformed modular group, was defined by Ren and Yanagawa [RY25]. Thanks to this tool, they gave other proofs for the arithmetic results of [KMR+25], and extended them to the left version of q-rational numbers.

The q-deformed modular group can be understood as a faithful representation of $\mathrm{PSL}_2(\mathbb{Z})$ in $\mathrm{PGL}_2(\mathbb{Z}[q, q^{-1}])$. In Chapter 1, we will see that this is the only such representation, up to conjugacy. If one allows complex coefficients, then the uniqueness fails, as shown by Topkara and Uludağ in [TU25]. They find in particular two other representations, working over the ring $\mathbb{Z}[\sigma][q, q^{-1}]$, with σ a primitive 6th root of unity.

Homological algebra interpretations

We already presented the work of Bapat, Becker and Licata as the first study of the left version of q-rational numbers. But the main focus of their paper [BBL23] relates to the description of compactifications of the space of stability conditions of a category of type A_2. These compactifications are parametrized by a positive real number q, and for each value of q, they prove that there is a homomorphism ρ_q between the boundary of the q-compactified space and the boundary of the q-Farey tessellation in the hyperbolic plane.

In the course of the proof, they q-deform an already known bijection between spherical objects of the category and rational numbers. Given a spherical object X associated to the rational number x, they consider algebraic data attached to X and compute from this the preimage by ρ_q of the geodesic between $[x]_q^\flat$ and $[x]_q^\sharp$ in the boundary of the q-Farey tessellation. Moreover, there is an action of the braid group B_3 on the compactification space. They show that the homomorphism ρ_q intertwines this action and the q-deformed action of the modular group on the q-Farey tessellation.

The combinatorial aspects of this work were investigated by Ren in [Ren24], who studied the interpretation of the q-Farey sum in the homological algebra context, and gave a new proof of a result of [BBL23] about left q-rationals using only combinatorial methods.

This homological algebra framework was also studied by Fan, Qiu, who built a topological realization of q-deformations, showing that a q-rational number can be computed using a notion of q-intersection of arcs in a bigraded decorated disk [FQ25]. In addition, they consider a categorification of their topological model, recovering the results of Bapat, Becker and Licata, and embedding them in a more general setting.

Summary of the thesis

The general aim of this thesis is extending the construction defining q-real numbers of Morier-Genoud and Ovsienko, and exploring the links between this theory and other mathematical fields. Several directions were investigated, corresponding to several points of view on q-numbers.

The first direction one focuses on underlying symmetries of the quantization, carried by the action of the modular group. We define in Chapter 1 a double extension of this action, and we deduce from it new relations between the two versions of q-rational numbers (left and right). By understanding better the symmetries of q-real numbers, we are able to study the behaviour of some q-deformed algebraic numbers, in the continuation of the previous works on quadratic and cubic numbers.

The second direction we consider is combinatorial. From the deformation of rational numbers, viewed as a set of numbers embedded in a combinatorial framework, we can deform other subsets of numbers in other combinatorial framework. This is the same principle behind the deformation of factorials and binomial coefficients, built from the q-integers but different from them. In Chapter 2, we present a quantization of Markov numbers based on traces of q-deformed matrices attached to q-rational numbers. The main results of this chapter come from a collaboration with Sam Evans, Sophie Morier-Genoud and Valentin Ovsienko.

The third axis deals with the geometry of q-real numbers. The initial motivation of this work was the extension of the deformation map from real numbers to complex numbers, which seems to be difficult to achieve with algebraic techniques. In Chapter 3, we thus introduce a geometric interpretation of q-real numbers, in the hope that this approach will provide a well-designed background for a future deformation of the complex plane. For the moment, our work only covers the real line, where a q-rational number is replaced by a disk depending on the real parameter q. One major contribution of this point of view is the description of new operations on q-rational numbers, that we call Springborn operations. This chapter is entirely based on a common work with Olga Paris-Romaskevich and Alexander Thomas, who initialised the project.

The fourth and last topic of the thesis is the generalization of q-number theory in higher dimension. We choose in Chapter 4 to use the correspondence between quantum analogues of numbers and the Burau representation of the braid group B_3 to define a deformation of the rational plane from the Burau representation of B_4. To this end, we classify orbits of the plane under the action of B_4, and we describe stabilizers of this same action. The q-deformed plane that results from this is particularly rich, and would need more investigation to understand its structure.

More details on the content of the chapters can be found at the beginning of each of them. Part of the works presented in this thesis have been released in the following articles.

[Jou25b] P. Jouteur. Symmetries of the q-deformed real projective line. *arXiv*, 2503.02122, 2025.

[EJMGO26] S. Evans, P. Jouteur, S. Morier-Genoud, V. Ovsienko. On q-deformed Markov numbers. Cohn matrices and perfect matchings with weighted edges. *Annals of Combinatorics*, 2026.

[JPRT26] P. Jouteur, O. Paris-Romaskevich, A. Thomas. Plane geometry of q-rationals and Springborn operations. *arXiv*, 2603.04295, 2026.

[Jou25a] P. Jouteur. Burau representation of B_4 and quantization of the rational projective plane. *Comptes Rendus. Mathématiques,* 363:89-107, 2025.

When no reference is given for a result, it means either that this result is very classic and then we had previously referred to a textbook containing the result, either that the result had not been published before this thesis.

Chapter 1

Quantum analogues of numbers old and new

Analogues quantiques de nombres
réels.
Tourne l'algèbre qu'on amasse en
disques.

Permutation opportune

This chapter introduces the main objects of this thesis, q-deformed real numbers. Except for the last section, it will only deal with rational numbers, and thus begins by Section 1.1 with reminders on the very classical combinatorics of rational numbers. Their q-analogues will be constructed in Section 1.2. Then Section 1.3 will provide a broader framework to handle q-rational numbers. In Section 1.4, we present q-deformations of the combinatorics introduced in the first section. Finally, Section 1.5 is dedicated to the construction of q-irrational numbers, with a particular focus on algebraic numbers.

The first and second sections are not new, they follow the existing literature on q - rationals (mainly [MGO20, LMG21]), except for §1.2.4 on uniqueness of q-rational numbers. Sections 1.3 and 1.5 are adapted from the article [Jou25b]. The content of Section 1.4 is not new, except the material related to q-matrices of determinant -1 which comes from [Jou25b], and §1.4.3, based on a joint work with Olga Paris-Romaskevich and Alexander Thomas [JPRT26].

Main results of Chapter 1. the main contribution of the chapter is the extension of the q-deformed action of the modular group on q-real numbers, contained in Section 1.3. In particular,

- Theorem 1.3.6 describes an equivariant action of $\mathrm{PGL}_2(\mathbb{Z})$ on the q-real line,
- Proposition 1.3.15 describes an equivariant action of an order 2 extension of $\mathrm{PGL}_2(\mathbb{Z})$.

Another important result is the total positivity for mixed left and right q-rational numbers.

- Theorem 1.4.33 establishes the total positivity for pairs of q-rational numbers in either left or right version.

Contents

1.1 Background on the combinatorics of rational numbers

1.1.1 Continued fractions

We refer to [Kar22] or [HW08] for complete expositions on continued fractions and rational numbers.

Definition 1.1.1. Let x be a rational number. A *continued fraction expansion* of x is a sequence of integers $a_1, a_2, \cdots, a_n$, such that

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}. \tag{1.1}$$

Two canonical continued fraction expansions of rational numbers will be of particular interest for us.

Proposition 1.1.2. *Every rational number x admits a positive continued fraction expansion and a negative continued fraction expansion,*

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_k}}}, \tag{1.2}$$

with $a_i \geq 1$ for $i \geq 2$, and $c_j \geq 2$ for $j \geq 2$.
The positive continued fraction expansion is unique provided the parity of n is fixed. The negative continued fraction expansion is unique.

Notation 1.1.3. Let x be a rational number. We will denote by

$$x = [a_1, a_2, \cdots, a_n]$$

the even or odd positive continued fraction expansion of x (the parity will always be specified). It is also called the *regular continued fraction* of x.
We denote by

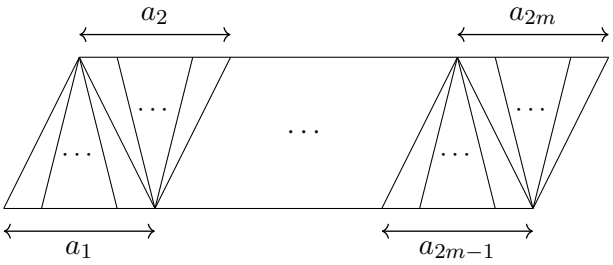
$$x = [\![c_1, c_2, \cdots, c_k]\!]$$

the negative continued fraction expansion of x. It is also called the *Hirzebruch continued fraction* of x.

Remark 1.1.4. We will often consider $\infty = \frac{1}{0}$ as a rational number, and we have $[\ ] = [\![\ ]\!] = \frac{1}{0}$.

The positive and negative continued fraction expansions of a rational number carry a rich combinatorics, detailed in [MGO19].

Definition 1.1.5. Let $x = [a_1, a_2, \cdots, a_{2m}]$ be a positive rational number. The *triangulated polygon* $\mathbb{T}_x$ associated to x is the polygon with $a_1 + a_2 + \cdots + a_{2m} + 2$ vertices, with a_1 triangles down-based, then a_2 triangles up-based, and so on.

$\mathbb{T}_x =$ 

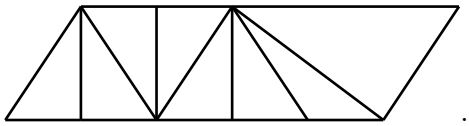
In the triangulated polygon $\mathbb{T}_x$, denote by $c_1, c_2, \cdots, c_k$ the *quiddity sequence* of the upper side from left to right. In other words, label the vertices of the upper side 1, 2, $\cdots$, until the one just before the end of the upper side, and then c_i is the number of triangles incident to the ith vertex.

Remark 1.1.6. In the definition above, if $0 < x < 1$, then the first coefficient of the regular continued fraction is zero. In this case, the polygon $\mathbb{T}_x$ begins with a_2 up-based triangles.
By convention $\mathbb{T}_{\frac{1}{0}}$ is empty.

Proposition 1.1.7 ([Hir73, HZ77]). *Let x be a positive rational number, with canonical continued fraction expansions $x = [a_1, a_2, \cdots, a_{2m}] = [\![c_1, c_2, \cdots, c_k]\!]$. Then the quiddity sequence of $\mathbb{T}_x$ is exactly $(c_1, c_2, \cdots, c_k)$.*
In particular, $k = a_2 + a_4 + \cdots + a_{2m}$, and

$$(c_1, c_2, \ldots, c_k) = (a_1 + 1, \underbrace{2, \ldots, 2}_{a_2 - 1}, a_3 + 2, \underbrace{2, \ldots, 2}_{a_4 - 1}, \cdots, a_{2m-1} + 2, \underbrace{2, \ldots, 2}_{a_{2m} - 1}).$$

Example 1.1.8. Let $x = \frac{22}{9}$. Its even positive continued fraction expansion is $[2, 2, 3, 1]$, and the negative one is $[\![3, 2, 5]\!]$. The triangulated polygon $\mathbb{T}_{\frac{22}{9}}$ looks like



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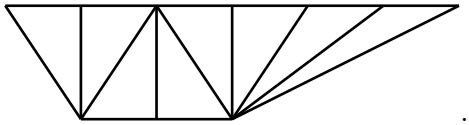
Example 1.1.9. Similarly, for $x = \frac{9}{22} = [0, 2, 2, 4] = [\![1, 2, 4, 2, 2, 2]\!]$, the polygon $\mathbb{T}_{\frac{9}{22}}$ is



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1.1.2 Farey graph

This subsection is based on [MGO19]. By convention, a rational number in reduced form is a fraction $\frac{a}{b}$ with a, b two coprime integers, and $b > 0$.

Definition 1.1.10. Given two rational numbers in reduced form $\frac{a}{b}$ and $\frac{c}{d}$, their *Farey determinant* is

$$d_F\left(\frac{a}{b}, \frac{c}{d}\right) = |ad - bc|.$$

The *Farey sum* of the pair is

$$\frac{a}{b} \oplus_F \frac{c}{d} = \frac{a + c}{b + d}.$$

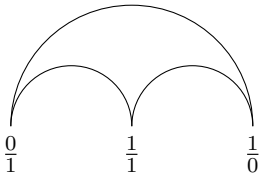
We extend this definition to $\mathbb{P}^1(\mathbb{Q})$ by allowing $\frac{1}{0}$ in the formulas.

Lemma 1.1.11. *Let $x, y \in \mathbb{P}^1(\mathbb{Q})$. If the pair (x, y) has Farey determinant 1, then the pairs $(x, x \oplus_F y)$ and $(x \oplus_F y, y)$ also have Farey determinant 1.*

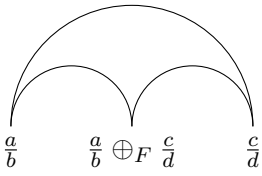
Remark 1.1.12. The Farey sum does not preserve the Farey determinant in general. For example, the pair $(1, 3)$ has Farey determinant 2, but the Farey sum is $1 \oplus_F 3 = 4$ and $(1, 4)$ has Farey determinant 3.

Definition 1.1.13. The *Farey graph* is a non oriented graph, whose vertices are the nonnegative rational numbers and $\frac{1}{0}$, and such that two numbers x and y are linked if and only if they have Farey determinant 1.

The Farey graph can be constructed recursively, starting with



and following the local rule



Extended to the negative rational numbers, the Farey graph realizes a tessellation of the hyperbolic plane by ideal triangles, we will use this in Chapter 3.

Definition 1.1.14. Let $0 < z < \frac{1}{0}$ be a rational number. There is a unique pair $x < y$ of numbers in $\mathbb{P}^1(\mathbb{Q})$ with $d_F(x, y) = 1$ such that $x \oplus_F y = z$. We say that x (resp. y) is the *left* (resp. *right*) *parent* of z, and that z is the *child* of the pair (x, y).

Notation 1.1.15. For any nonnegative rational number x, there is a minimal subgraph of the Farey graph containing x, denoted by $\mathbb{T}_x^F$.

The positive and negative continued fractions of x can be read on the graph $\mathbb{T}_x^F$.

Proposition 1.1.16. *Let x be a nonnegative rational number and $\mathbb{T}_x^F$ its Farey subgraph.*

1. *Let $[a_1, a_2, \cdots, a_n]$ be the (odd or even) positive continued fraction expansion of x. Then, following the vertical above x from top to bottom, it crosses a_1 triangles based on the left, then a_2 triangles based on the right, and so on until reaching x.*

2. *Similarly, let $[\![c_1, c_2, \cdots, c_k]\!]$ be the negative continued fraction expansion of x. Then, following the horizontal segment from $\frac{1}{0}$ to x, it crosses vertices of $\mathbb{T}_x^F$. The first vertex is incident to c_1 triangles, the second is incident to c_2 triangles, and so on until the right neighbour of x.*

Example 1.1.17. Take $x = 22/9$. In the picture below, we can read the expansion $22/9 = [2, 2, 3, 1]$ on the red line from top to bottom, and $22/9 = [\![3, 2, 5]\!]$ in blue.

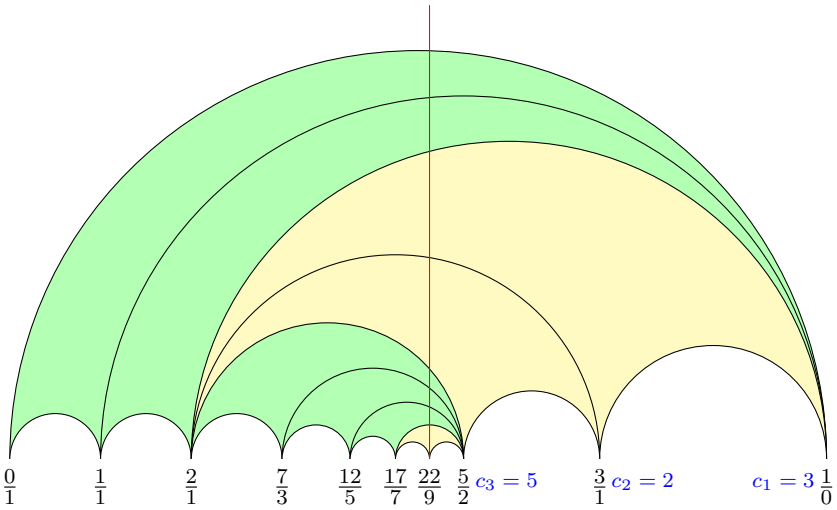
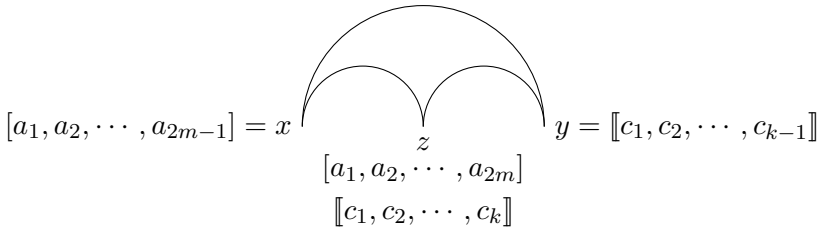


Figure 1.1: Subgraph $\mathbb{T}^F_{22/9}$ of the Farey graph

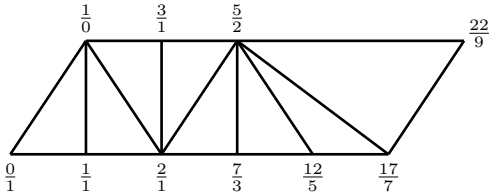
Given a rational number, its two parents in the Farey graph can be computed using continued fractions expansions.

Corollary 1.1.18. *Let z be a positive rational number ($z \neq \frac{1}{0}$), and let $x < y$ be its two parents. Suppose z has even positive continued fraction $[a_1, a_2, \cdots, a_{2m}]$ and negative continued fraction $[\![c_1, c_2, \cdots, c_k]\!]$. Then $x = [a_1, a_2, \cdots, a_{2m-1}]$ and $y = [\![c_1, c_2, \cdots, c_{k-1}]\!]$.*

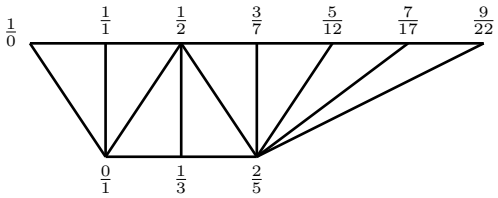


The relationship between continued fractions and the Farey graph actually comes from the fact that for any positive rational number x, the subgraph $\mathbb{T}^F_x$ of the Farey graph is isomorphic to the triangulated polygon $\mathbb{T}_x$ associated to x. Hence, one can construct $\mathbb{T}_x$ by a concatenation of triangles dictated by the continued fraction expansions of x, and the vertices of $\mathbb{T}_x$ can be labelled with the genealogy of x in the Farey graph. The right-most vertex of a triangle in $\mathbb{T}_x$ is labelled with the Farey sum of the labels of the two other vertices of the triangle.

Example 1.1.19. Let us continue with the example $x = \frac{22}{9}$. The triangulated polygon $\mathbb{T}_x$ can be labelled with the genealogic tree of $\frac{22}{9}$.



For a rational number lower than 1, it works similarly, with the labels of the first triangle rotated.



1.1.3 Modular group

Consider the following group of 2×2 matrices, called the modular group.

$$\mathrm{PSL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\} / \pm \mathrm{Id}\,. \tag{1.3}$$

It is a classical result [Ser73] that this group has a presentation with two generators and two relations, $\mathrm{PSL}_2(\mathbb{Z}) = \langle T, S \mid S^2 = (TS)^3 = 1 \rangle$, with

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.4}$$

Definition 1.1.20. The group $\mathrm{PSL}_2(\mathbb{Z})$ acts on the projective rational line $\mathbb{P}^1(\mathbb{Q})$, by *fractional linear transformations*.

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot x = \frac{ax + c}{bx + d}.$$

Lemma 1.1.21. *This action* $\mathrm{PSL}_2(\mathbb{Z}) \curvearrowright \mathbb{P}^1(\mathbb{Q})$ *is transitive.*

Proof. Let $\frac{a}{b}$ be a rational number in reduced form. As a and b are coprime, there exists integers x, y such that the Bézout relation holds : $ay - bx = 1$. Then the matrix $M = \begin{pmatrix} a & x \\ b & y \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ sends $\frac{1}{0}$ on $\frac{a}{b}$. □

The continued fractions theory provides two ways of constructing matrices reaching a given rational number from $\frac{1}{0}$. To this end, following [MGO19], consider the matrix

$$L = TST = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{1.5}$$

Definition 1.1.22. Let x be a rational number and let $x = [a_1, a_2, \cdots, a_{2m}]$ be its even positive continued fraction expansion. Define

$$M_+(x) = T^{a_1} L^{a_2} \cdots T^{a_{n-1}} L^{a_{2m}}. \tag{1.6}$$

Similarly, let $x = [\![c_1, c_2, \cdots, c_k]\!]$ be the negative continued fraction of x. Define

$$M_-(x) = T^{c_1} S T^{c_2} S \cdots T^{c_k} S. \tag{1.7}$$

Proposition 1.1.23. *Let x be a rational number. Then $M_+(x)\cdot\frac{1}{0}=x$ and $M_-(x)\cdot\frac{1}{0}=x$. Moreover, if $x=\frac{a}{b}$ in reduced form, and $x=[a_1,a_2,\cdots,a_{2m}]=[\![c_1,c_2,\cdots,c_k]\!]$, then*

$$M_+(x)=\begin{pmatrix}a & a'\\ b & b'\end{pmatrix} \text{ and } M_-(x)=\begin{pmatrix}a & -a''\\ b & -b''\end{pmatrix},$$

with $\frac{a'}{b'}=[a_1,a_2,\cdots,a_{2m-1}]$ and $\frac{a''}{b''}=[\![c_1,c_2,\cdots,c_{k-1}]\!]$, the two parents of x.

Remark 1.1.24. For $x=\frac{1}{0}$, $M_+\left(\frac{1}{0}\right)=M_-\left(\frac{1}{0}\right)=\mathrm{Id}$.

Example 1.1.25. With $x=\frac{22}{9}=[2,2,3,1]=[\![3,2,5]\!]$, we get

$$M_+\begin{pmatrix}22\\9\end{pmatrix}=\begin{pmatrix}22 & 17\\ 9 & 7\end{pmatrix} \text{ and } M_-\begin{pmatrix}22\\9\end{pmatrix}=\begin{pmatrix}22 & -5\\ 9 & -2\end{pmatrix}.$$

1.2 First approach of q-rational numbers via matrices

Historically, q-rational numbers were defined by Morier-Genoud and Ovsienko [MGO20] using q-deformed continued fraction expansions. Later, in [LMG21], a more algebraic point of view was given, using an equivariance property with respect to a q-deformed action of $\mathrm{PSL}_2(\mathbb{Z})$ on the rational projective line. We will follow this point of view, which highlights how the second version of q-rational numbers, the left one, arises naturally from the definitions [BBL23].

1.2.1 Definitions : projective groups of matrices

We begin with reminders on groups of matrices and their projective versions. Let R be an integral domain, and U_R the set of units of R.

Definition 1.2.1. The *general linear group* $\mathrm{GL}_2(R)$ is the group of 2×2 matrices with coefficients in R and such that the determinant is in U_R.
The *projective linear group* $\mathrm{PGL}_2(R)$ is the quotient $\mathrm{GL}_2(R)/Z(\mathrm{GL}_2(R))$, and the center $Z(\mathrm{GL}_2(R))$ consists of the scalar transformations $u\,\mathrm{Id}$ for $u\in U_R$.

When working with an element A of $\mathrm{PGL}_2(R)$, we will consider a representant of A in $\mathrm{GL}_2(R)$, and by abuse of notation we will also denote this representant by A.

Definition 1.2.2. Denote by $\overline{R}$ the field of fractions of R. The projective linear group $\mathrm{PGL}_2(R)$ acts by fractional linear transformations on $\mathbb{P}^1(\overline{R})$.

$$\begin{array}{ccc}\mathrm{PGL}_2(R)\times\mathbb{P}^1(\overline{R}) & \longrightarrow & \mathbb{P}^1(\overline{R})\\ \begin{pmatrix}a & b\\ c & d\end{pmatrix},[x:y] & \longmapsto & [ax+by:cx+dy]\end{array}.$$

Notation 1.2.3. Elements of $\mathbb{P}^1(\overline{R})$ will often be denoted as reduced fractions $\frac{a}{b}$, with a and b in R, and in particular the point $[1:0]$ is denoted $\frac{1}{0}$.

1.2.2 A q-version of the modular group

From now on, q denotes a formal indeterminate.

Definition 1.2.4 ([MGO20, LMG21])**.** Consider the two q-deformed generators of the modular group,

$$T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}[q, q^{-1}]). \tag{1.8}$$

Denote by $\mathrm{PSL}_{2,q}(\mathbb{Z})$ the subgroup of $\mathrm{PGL}_2(\mathbb{Z}[q, q^{-1}])$ generated by (the class of) T_q and (the class of) S_q.

Remark 1.2.5. The q-matrices T_q and S_q satisfy, up to powers of q and sign, the defining relations of the modular group:

$$S_q^2 = -q\,\mathrm{Id} \text{ and } (T_q S_q)^3 = -q^3\,\mathrm{Id}\,.$$

In what follows, unless explicitely mentionned, the q-matrices will be considered up to sign and power of q.

Proposition 1.2.6 ([LMG21])**.** *There is a well-defined group isomorphism*

$$\begin{array}{ccc} \mathrm{PSL}_2(\mathbb{Z}) & \xrightarrow{\simeq} & \mathrm{PSL}_{2,q}(\mathbb{Z}) \\ M & \longmapsto & M_q \end{array}, \tag{1.9}$$

where M_q is obtained by replacing each T by T_q and each S by S_q in a word in T, S representing M. The inverse of this isomorphism is given by the evaluation $q = 1$.

Example 1.2.7. Via the isomorphim $\mathrm{PSL}_{2,q}(\mathbb{Z}) \simeq \mathrm{PSL}_2(\mathbb{Z})$, the matrix $L = TST$ is q-deformed as

$$L_q = T_q S_q T_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}.$$

Remark 1.2.8. With a representation theoretic point of view, the construction of $\mathrm{PSL}_{2,q}(\mathbb{Z})$ actually corresponds to the projective version of the famous reduced Burau representation of the braid group B_3 [Bur35]. This correspondence will be exploited in Chapter 4.

$$\begin{aligned} \rho_3 : B_3 &\longrightarrow \mathrm{GL}_2(\mathbb{Z}[q, q^{-1}]) \\ \sigma_1 &\longmapsto \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} = T_q \\ \sigma_2 &\longmapsto \begin{pmatrix} 1 & 0 \\ -q & q \end{pmatrix} = L_q^{-1}. \end{aligned}$$

1.2.3 Definition of q-rational numbers

When the modular group was acting on the rational projective line, its q-version will act on rational fractions in q, again by fractional linear transformations. This q-action will define the q-deformed rational numbers.

To be more precise, the q-deformed rational line will be one of the orbits in $\mathbb{P}^1(\mathbb{Z}(q))$ under the action of $\mathrm{PSL}_{2,q}(\mathbb{Z})$. The following proposition states which orbit one can pick.

Proposition 1.2.9 ([Jou25b]). *The only two orbits of* $\mathrm{PSL}_{2,q}(\mathbb{Z})$ *in* $\mathbb{P}^1(\mathbb{Z}(q))$ *giving a well-defined q-deformed rational projective line are those of* $\frac{1}{0}$ *and* $\frac{1}{1-q}$.

Proof. Let us consider the specialization map

$$\begin{array}{rccc} \mathrm{ev}_1 : & \mathbb{P}^1(\mathbb{Z}(q)) & \longrightarrow & \mathbb{P}^1(\mathbb{Q}) \\ & f(q) & \longmapsto & f(1) \end{array}.$$

Because the following diagram is commutative and the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ is transitive, any orbit $\mathcal{O}$ of $\mathrm{PSL}_{2,q}(\mathbb{Z})$ maps by ev_1 onto $\mathbb{P}^1(\mathbb{Q})$ surjectively.

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{Z}(q)) & \xrightarrow{\mathrm{ev}_1} & \mathbb{P}^1(\mathbb{Q}) \\ \cup & \circlearrowleft & \cup \\ \mathrm{PSL}_{2,q}(\mathbb{Z}) & \simeq & \mathrm{PSL}_2(\mathbb{Z}) \end{array}$$

Thus an orbit $\mathcal{O}$ always contains at least one point f_∞ such that $f_\infty(1) = \frac{1}{0}$. And the orbit $\mathcal{O}$ defines a deformation of the rational projective line if and only if ev_1 is injective on it, which amounts to saying that the stabilizer of f_∞ is isomorphic to the stabilizer of $\frac{1}{0}$ in $\mathrm{PSL}_2(\mathbb{Z})$. The latter is the group generated by T.
Therefore the orbit $\mathcal{O}$ is a q-deformation of the rational projective line if and only if f_∞ is one of the fixed point of T_q, and there are exactly two of them : $\frac{1}{0}$ and $\frac{1}{1-q}$. □

Now we can define the two versions of q-deformed rational numbers. The right version (which is the original one, defined in [MGO20]) corresponds to the orbit of $\frac{1}{0}$ under $\mathrm{PSL}_{2,q}(\mathbb{Z})$, and the left version (defined in [BBL23]) corresponds to the orbit of $\frac{1}{1-q}$.

Definition 1.2.10. Let $x \in \mathbb{Q}$ be a rational number and let $M \in \mathrm{PSL}_2(\mathbb{Z})$ be such that $M \cdot \frac{1}{0} = x$. Then the two deformations of x are elements of $\mathbb{P}^1(\mathbb{Z}(q))$, given by

- $[x]_q^\sharp = M_q \cdot \frac{1}{0}$ (*right version*),
- $[x]_q^\flat = M_q \cdot \frac{1}{1-q}$ (*left version*).

Remark 1.2.11. The definitions of $[x]_q^\sharp$ and $[x]_q^\flat$ do not depend on the choice of the matrix M sending $\frac{1}{0}$ to x. This is a consequence of Proposition 1.2.9.

In the rest of the document, unless when necessary, we will skip the subscript "q" in the notation of q-rational numbers, writing $[x]^\sharp$ instead of $[x]_q^\sharp$ and $[x]^\flat$ instead of $[x]_q^\flat$.

Example 1.2.12. The right version recovers the usual q-integers : for $n \in \mathbb{Z}_{\geq 0}$,

$$[n]^\sharp = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}. \tag{1.10}$$

The left version produces different q-analogues of integers.

$$[n]^\flat = \frac{q^{n+1} - q^n + q^{n-1} - 1}{q - 1} = 1 + q + \cdots + q^{n-2} + q^n. \tag{1.11}$$

Remark 1.2.13. The terms "left" and "right" versions refer to the behaviour of sequences of q-rational numbers when converging to a rational number from the right or from the left. More details will be given in Section 1.5.

As a direct consequence of Definition 1.2.10, we have the equivariance of the quantization maps.

Corollary 1.2.14 ([LMG21, BBL23]). *Let $M \in \mathrm{PSL}_2(\mathbb{Z})$, let $x \in \mathbb{P}^1(\mathbb{Q})$. Then*

$$M_q \cdot [x]^\sharp = [M \cdot x]^\sharp, \text{ and } M_q \cdot [x]^\flat = [M \cdot x]^\flat.$$

Example 1.2.15. We can deduce from the equivariance property the q-analogues of inverses of integers : for $n \in \mathbb{Z}_{>0}$,

$$\left[\frac{1}{n}\right]^\sharp = S_q \cdot [-n]^\sharp = \frac{q^{n-1}}{1+q+\cdots+q^{n-1}},$$

$$\left[\frac{1}{n}\right]^\flat = S_q \cdot [-n]^\flat = \frac{q^{n}}{1+q^2+\cdots+q^{n}}.$$

We will find in Section 1.3 a direct formula to compute the analogues of the inverse of a rational number directly (without needing to use negative numbers as intermediate steps).

Example 1.2.16. The q-rational numbers are not quotient of q-integers. For example,

$$\left[\frac{5}{2}\right]^\sharp = T_q^2 L_q^2 \cdot \frac{1}{0} = \frac{1+2q+q^2+q^3}{1+q}.$$

1.2.4 Discussion on the uniqueness of q-rational numbers

The construction of q-rational numbers described above relies on a limited number of choices, chief among them being the choice of a faithful representation of $\mathrm{PSL}_2(\mathbb{Z})$ in $\mathrm{PGL}_2(\mathbb{Z}[q,q^{-1}])$. In this subsection, we will discuss these choices and explain in what sense left and right q-rationals are essentially the only natural ones.

Our first assumption is that q-rational numbers should have a nice geometric behaviour with respect to fractional linear transformations.

Assumption 1 : the quantization map is equivariant through the action of $\mathrm{PSL}_2(\mathbb{Z})$.

The second step is the choice of the q-analogue of the matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Indeed, already for the definition of the usual q-integers, one has to choose how the shift $n \mapsto n+1$ is being deformed. Euler and Gauss q-integers appear in a wide range of mathematical areas, in combinatorics, representation theory of Lie algebras, hypergeometric series... In all cases, the identity $[n+1]_q = q[n]_q + 1$ plays a crucial role. This provides strong motivation to define $T_q : [n]_q \mapsto q[n]_q + 1$, and we will take this choice as granted.

$$\text{Assumption 2 : } T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}.$$

Now that T_q is fixed, it only remains to choose a q-analogue of the second generator S to get a representation of $\mathrm{PSL}_2(\mathbb{Z})$. We must discuss here the choice of the ring of coefficients for q-matrices. The smallest ring compatible with the definition of T_q is the ring of Laurent polynomials $\mathbb{Z}[q,q^{-1}]$. We then choose to work in $\mathrm{PGL}_2(\mathbb{Z}[q,q^{-1}])$ from now on.

$$\text{Assumption 3 : the base ring is } \mathbb{Z}[q,q^{-1}].$$

It turns out that there are then only two options for a q-version of S, and they are conjugated in $\mathrm{PGL}_2(\mathbb{Z}(q))$. A more general version of this result will appear in an incoming work with Olga Paris-Romaskevich and Alexander Thomas. Some work has also been done in this direction in [TU25], where the authors allow complex coefficients.

Theorem 1.2.17. *Let $\rho : \mathrm{PSL}_2(\mathbb{Z}) \longrightarrow \mathrm{PGL}_2(\mathbb{Z}[q, q^{-1}])$ be a representation, such that for all $M \in \mathrm{PSL}_2(\mathbb{Z})$, $\rho(M)_{|q=1} = M$. Suppose $\rho(T) = T_q$. Then $\rho(S)$ is one of the two following matrices*

$$S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} \ , \ \widehat{S_q} = \begin{pmatrix} 1-q & -1 \\ q^2-q+1 & q-1 \end{pmatrix}.$$

Moreover, the two representations are conjugated in $\mathrm{PGL}_2(\mathbb{Z}(q))$, by

$$U_q = \begin{pmatrix} q & 1-q \\ 0 & q^2-q+1 \end{pmatrix}.$$

Proof. As ρ is a representation of $\mathrm{PSL}_2(\mathbb{Z})$, the relations $(\rho(T)\rho(S))^3 = 1$ and $\rho(S)^2 = 1$ must hold. By hypothesis, $\rho(T) = T_q$. Let us write $\rho(S) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. As an element of $\mathrm{PGL}_2(\mathbb{Z}[q, q^{-1}])$, this matrix has determinant $\pm q^k$, for $k \in \mathbb{Z}$. Up to multiplication by $\pm q$, we can fix a representative of $\rho(S)$ whose determinant is $ad - bc \in \{\pm 1, \pm q\}$.
The first relation $\rho(S)^2 = 1$ implies that $d = -a$. If $a = 0$, then one can check by direct computations that the only solution is the usual $S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$.
Suppose $a \neq 0$. As $a(1) = 0$, there is $\tilde{a} \in \mathbb{Z}[q, q^{-1}]$ such that $a = (q-1)\tilde{a}$. Then the relation $(T_q\rho(S))^3 = 1$ implies the expression of b as a function of c and $\tilde{a}$:

$$b = -\frac{\tilde{a}}{qc}\left(\frac{c^2}{\tilde{a}} + (q-1)^2((q^2-q+1)\tilde{a} + 2c)\right).$$

With this expression, $\det(\rho(S)) = \frac{1}{q}(c + \tilde{a}(q-1)^2)^2$. But we chose a representative such that $\det(\rho(S)) \in \{\pm 1, \pm q\}$, thus it must be $+q$, and

$$(c + \tilde{a}(q-1)^2)^2 = q^2, \text{ so } c = \pm q - \tilde{a}(q-1)^2.$$

Without loss of generality, let us chose $c = q - \tilde{a}(q-1)^2$. For the coefficient b to be a Laurent polynomial, c must divide $(q^2-q+1)\tilde{a}$, and this implies that $\tilde{a} = -1$. In the end,

$$\rho(S) = \begin{pmatrix} 1-q & -1 \\ q^2-q+1 & q-1 \end{pmatrix}.$$

Finally, one can check that U_q commutes with T_q and satisfies $U_q S_q U_q^{-1} = \widehat{S_q}$. □

By the previous Theorem, one could define another q-version of the modular group, $\widehat{\mathrm{PSL}_2(\mathbb{Z})}$ as the image of the representation given by $\widehat{S_q}$. With this new q-action, one could define a hat version of q-rational numbers, i.e. for any $x \in \mathbb{Q}$ and any $M \in \mathrm{PSL}_2(\mathbb{Z})$ with $M \cdot \infty = x$,

$$\widehat{[x]}^\sharp = \widehat{M_q} \cdot \frac{1}{0} \ , \ \widehat{[x]}^\flat = \widehat{M_q} \cdot \frac{1}{1-q}.$$

Notice that the matrix U_q stabilises the rational fractions $\frac{1}{0}$ and $\frac{1}{1-q}$. This means that the hat q-rational numbers are actually the images by U_q of usual q-rational numbers.

$$\widehat{[x]}^\sharp = U_q \cdot [x]^\sharp, \text{ and } \widehat{[x]}^\flat = U_q \cdot [x]^\flat.$$

In particular, $U_q \cdot [0]^\flat = [0]^\sharp$, so the left-hat-$q$-integers $\widehat{[n]}^\flat$ are actually the well known Euler and Gauss q-integers . This means that one recovers Euler and Gauss q-integers in the two settings, with S_q and $\widehat{S_q}$. Finally, the choice made by Morier-Genoud and Ovsienko amounts to adding the last assumption

Assumption 4 : the analogue of $\frac{0}{1}$ is $\frac{0}{1}$ and the analogue of $\frac{1}{0}$ is $\frac{1}{0}$.

1.3 Extensions of symmetries

This section is an adapted version of [Jou25b]. One of the motivation of this work is to unify left and right q-rational numbers into one single orbit, by extending the group of their symmetries. The extension we consider proceeds in two steps. First, we define a q-deformed action of $\mathrm{PGL}_2(\mathbb{Z})$ generalizing the q-version of the modular group $\mathrm{PSL}_{2,q}(\mathbb{Z})$. Second, we twist the q-deformed action of $\mathrm{PGL}_2(\mathbb{Z})$ by a central extension.

1.3.1 From $\mathrm{PSL}_2(\mathbb{Z})$ to $\mathrm{PGL}_2(\mathbb{Z})$

The group $\mathrm{PGL}_2(\mathbb{Z})$ admits a presentation by three generators, compatible with the one of $\mathrm{PSL}_2(\mathbb{Z})$ given above (1.4).

$$\mathrm{PGL}_2(\mathbb{Z}) = \langle T, S, N \mid S^2 = (TS)^3 = N^2 = (TN)^2 = (SN)^2 = 1 \rangle, \tag{1.12}$$

with $N = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

In order to properly introduce a q-deformed version of the transformation $N : x \mapsto -x$, we need to work on a slightly bigger ring of coefficients.

Notation 1.3.1. Let $t_q = q^2 - q + 1$, and Λ be the localization of $\mathbb{Z}[q, q^{-1}]$ by t_q.

$$\Lambda := \mathbb{Z}[q, q^{-1}][t_q^{-1}]. \tag{1.13}$$

The ring Λ is an integral domain, whose field of fractions is $\overline{\Lambda} = \mathbb{Z}(q)$, and whose units are $\{\pm q^i t_q^j \mid i, j \in \mathbb{Z}\}$.

It is worth noting that this polynomial t_q already played a role in Theorem 1.2.17, as the determinant of the matrix U_q bridging usual q-rational numbers with the hat version of them.

Definition 1.3.2. We define $N_q = \begin{pmatrix} -1 & 1-q^{-1} \\ q-1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\Lambda)$.
Then denote by $\mathrm{PGL}_{2,q}(\mathbb{Z})$ the subgroup of $\mathrm{PGL}_2(\Lambda)$ generated by (classes of) T_q, S_q and N_q.

Proposition 1.3.3 ([Jou25b])**.** *The group* $\mathrm{PGL}_{2,q}(\mathbb{Z})$ *is isomorphic to* $\mathrm{PGL}_2(\mathbb{Z})$.

Proof. Straightforward computations show that matrices T_q, S_q and N_q satisfy

$$N_q^2 = (N_qT_q)^2 = (N_qS_q)^2 = t_q \,\mathrm{Id}\,.$$

This enables to define a group homomorphism $[\cdot]_q : \mathrm{PGL}_2(\mathbb{Z}) \to \mathrm{PGL}_2(\Lambda)$, mapping T to T_q, S to S_q and N to N_q. By definition, the image of $[\cdot]_q$ is $\mathrm{PGL}_{2,q}(\mathbb{Z})$.
Now, if we denote by ev_1 the map $\mathrm{PGL}_{2,q}(\mathbb{Z}) \longrightarrow \mathrm{PGL}_2(\mathbb{Z})$ given by the evaluation $q = 1$, we have $\mathrm{ev}_1 \circ [\cdot]_q = \mathrm{id}$. Indeed, for each generator T, S and N this is true, and this extends to the whole group. Therefore $[\cdot]_q$ is injective. □

Example 1.3.4. We then have a q-deformation of the matrix $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We will use it to give a q-deformation of the operation $x \mapsto 1/x$.

$$J = NS \xrightarrow{q} J_q = N_qS_q = \begin{pmatrix} q-1 & 1 \\ q & 1-q \end{pmatrix}.$$

Note that we can choose T and J to be generators of $\mathrm{PGL}_2(\mathbb{Z})$, as we have the following relations

$$T^{-1}JTJT^{-1} = S \text{ and } JTJT^{-1}JT = N.$$

In the quantized setting, these relations become

$$T_q^{-1}J_qT_qJ_qT_q^{-1} = t_qS_q \text{ and } J_qT_qJ_qT_q^{-1}J_qT_q = t_qqN_q.$$

This q-version of $\mathrm{PGL}_2(\mathbb{Z})$ is the unique one compatible with $\mathrm{PSL}_{2,q}(\mathbb{Z})$.

Proposition 1.3.5. *Let $\rho : \mathrm{PGL}_2(\mathbb{Z}) \longrightarrow \mathrm{PGL}_2(\Lambda)$ be a representation, and suppose that $\rho(T) = T_q$ and $\rho(S) = S_q$. Then $\rho(N) = N_q$.*

Proof. Denote $\rho(N) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The relation $N^2 = \mathrm{Id}$ imposes that $a = -d$.
Let us compute $\rho(NT)$ and $\rho(NS)$.

$$\rho(NT) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} qa & a+b \\ qc & c-a \end{pmatrix}$$

$$\rho(NS) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} = \begin{pmatrix} qb & -a \\ -qa & -c \end{pmatrix}.$$

Then relations $(NT)^2 = \mathrm{Id}$ and $(NS)^2 = \mathrm{Id}$ in $\mathrm{PGL}_2(\mathbb{Z})$ implies that $aq = -c + a$ and $qb = c$, so

$$\rho(N) = \begin{pmatrix} a & a(q^{-1}-1) \\ a(1-q) & -a \end{pmatrix}.$$

Finally, $\det(\rho(N)) = -a^2(q^{-1} - 1 + q)$ must be a unit of Λ, so a is a unit of Λ, and we can normalize $\rho(N)$ to get N_q. □

The q-action of $\mathrm{PGL}_2(\mathbb{Z})$ exchanges left and right q-deformations of rational numbers.

Theorem 1.3.6 ([Jou25b])**.** *Let $M \in \mathrm{GL}_2(\mathbb{Z})$, with $\det(M) = -1$. Let $x \in \mathbb{P}^1(\mathbb{Q})$. Then*

$$M_q \cdot [x]^\sharp = [M \cdot x]^\flat \text{ and } M_q \cdot [x]^\flat = [M \cdot x]^\sharp.$$

Proof. Let us multiply M by the matrix N, so that $M = M^+N$ with $\det(M^+) = 1$, and $M_q = M_q^+ N_q$. We consider first the case $x = \infty$. Then

$$M_q \cdot [\infty]^\sharp = M_q^+ N_q \cdot \frac{1}{0} = M_q^+ \cdot \frac{1}{1-q} = [M^+ \cdot \infty]^\flat.$$

Now for any $x \in \mathbb{Q}$, there is a matrix $A \in \mathrm{PSL}_2(\mathbb{Z})$ such that $A \cdot \infty = x$, and then

$$M_q \cdot [x]^\sharp = M_q A_q \cdot [\infty]^\sharp = [MA \cdot \infty]^\flat = [M \cdot x]^\flat.$$

The equality $M_q \cdot [x]^\flat = [M \cdot x]^\sharp$ is symmetric. □

Corollary 1.3.7 ([Jou25b])**.** *Let $x \in \mathbb{P}^1(\mathbb{Q})$. Then*

$$[-x]^\flat = \frac{-[x]^\sharp + 1 - q^{-1}}{(q-1)[x]^\sharp + 1}, \tag{1.14}$$

$$\left[\frac{1}{x}\right]^\flat = \frac{(q-1)[x]^\sharp + 1}{q[x]^\sharp + 1 - q}, \tag{1.15}$$

and similarly with $\sharp$ and $\flat$ exchanged.

Because the q-matrices of determinant -1 are defined over the ring Λ, the polynomial t_q might come up when working with these matrices. To illustrate this, we give detailled computations in some examples.

Example 1.3.8. Let $n \in \mathbb{Z}_{>0}$ and let us compute the deformations of $\frac{1}{n}$ thanks to relation (1.15).

$$\begin{aligned}\left[\frac{1}{n}\right]^\sharp &= \frac{(q-1)[n]^\flat + 1}{q[n]^\flat + 1 - q} \\ &= \frac{q^{n+1} - q^n + q^{n-1}}{1 + q^2 + \cdots + q^{n-1} + q^{n+1}} \\ &= \frac{t_q q^{n-1}}{t_q(1 + q + \cdots + q^{n-1})}. \\ &= \frac{q^{n-1}}{[n]^\sharp}.\end{aligned}$$

On the other way around,

$$\begin{aligned}\left[\frac{1}{n}\right]^\flat &= \frac{(q-1)[n]^\sharp + 1}{q[n]^\sharp + 1 - q} \\ &= \frac{q^n}{1 + q^2 + \cdots + q^n}.\end{aligned}$$

Example 1.3.9. Let us consider now $x = \frac{3}{2}$ and $M = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$. Then M decomposes as $M = T^3 J$, so

$$M_q = T_q^3 J_q = \begin{pmatrix} q^4 + q^2 + q & 1 \\ q & 1 - q \end{pmatrix}.$$

One can compute $\left[\frac{3}{2}\right]^\flat = \frac{1+q^2+q^3}{1+q^2}$. Then applying the quantized matrix M_q and cancelling a factor t_q above and below the fraction,

$$M_q \cdot \left[\frac{3}{2}\right]^\flat = M_q \cdot \frac{1+q^2+q^3}{1+q^2} = \frac{q^5+2q^4+2q^3+3q^2+2q+1}{q^2+q+1}.$$

One can check that this coincide with $\left[M \cdot \frac{3}{2}\right]^\sharp = \left[\frac{11}{3}\right]^\sharp$.

1.3.2 Twisted q-action

Let us consider the operator of duality $q \leftrightarrow \frac{1}{q}$,

$$\begin{array}{rccc} \tau: & \mathbb{P}^1(\mathbb{Z}(q)) & \longrightarrow & \mathbb{P}^1(\mathbb{Z}(q)) \\ & f & \longrightarrow & f(q^{-1}) \end{array}.$$

Proposition 1.3.10 ([Jou25b])**.** *There is only one element I_q in $\mathrm{PGL}_2(\Lambda)$ such that the operator $I_q \circ \tau$ commutes with the quantized action of $\mathrm{PGL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Z}(q))$. It is given by the matrix*

$$\begin{pmatrix} 1 & q-1 \\ 1-q & q \end{pmatrix},$$

and it satisfies $(I_q \circ \tau)^2 = 1$.

Proof. Let I_q be given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that $I_q \cdot \tau$ commutes with every element A_q for $A \in \mathrm{PGL}_2(\mathbb{Z})$. In particular, $I_q \circ \tau \circ T_q = T_q \circ I_q \circ \tau$.
This means that up to a power of q,

$$\begin{pmatrix} aq^{-1} & a+b \\ cq^{-1} & c+d \end{pmatrix} = \begin{pmatrix} aq+c & qb+d \\ c & d \end{pmatrix}.$$

Putting $c = 0$ leads to $q = q^{-1}$, so we can suppose that $c \neq 0$.
Then the bottom-left coefficient imposes that the right hand side is q times the left hand side, and $\begin{cases} a = aq + c \\ qa + qb = qb + d \\ qc + qd = d \end{cases}$.
We deduce that $c = (1-q)a$, and $d = qa$.

$$I_q = \begin{pmatrix} a & b \\ (1-q)a & qa \end{pmatrix}.$$

Similarly with S_q, $I_q \circ \tau \circ S_q = S_q \circ I_q \circ \tau$, so up to a power of q,

$$\begin{pmatrix} b & -qa \\ qa & (q^2-q)a \end{pmatrix} = \begin{pmatrix} (1-q^{-1})a & -a \\ a & b \end{pmatrix},$$

hence $b = (q-1)a$. Finally, we can check that $I_q \circ \tau \circ N_q = N_q \circ I_q \circ \tau$.
To have $\det(I_q)$ invertible in Λ, the coefficient a must be invertible so a is a power of $t_q q$. □

Definition 1.3.11. We define $\overline{I}_q := I_q \circ \tau$ as the *twisted quantization* of the identity of $\mathbb{P}^1(\mathbb{Z}(q))$.
Then, for every matrix $M \in \mathrm{PGL}_2(\mathbb{Z})$, we define the operator

$$\overline{M}_q := M_q \circ \overline{I}_q \text{ acting on } \mathbb{P}^1(\mathbb{Z}(q)).$$

Remark 1.3.12. As $\overline{I}_q^2 = \mathrm{Id}$ in $\mathrm{PGL}_2(\Lambda)$, for all elements A, B in $\mathrm{PGL}_2(\mathbb{Z})$, we have

$$\overline{A}_q \overline{B}_q = A_q B_q.$$

Example 1.3.13. For the generators T, S and N, we get

$$\overline{T}_q = \begin{pmatrix} 1 & q^2 \\ 1-q & q \end{pmatrix} \circ \tau,\ \overline{S}_q = \begin{pmatrix} q-1 & -q \\ q & q^2-q \end{pmatrix} \circ \tau \text{ and } \overline{N}_q = \begin{pmatrix} -1 & 0 \\ 0 & q \end{pmatrix} \circ \tau.$$

Recall the other usual generator $L = TST$. In its twisted form, it becomes

$$\overline{T}_q \overline{S}_q \overline{T}_q = \begin{pmatrix} q & q^2-q \\ 1 & q^2 \end{pmatrix} \circ \tau = \overline{L}_q.$$

For the matrix $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the twisted deformation is trivial :

$$\overline{J_q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \tau.$$

Definition 1.3.14. We define the twisted q-deformation of $\mathrm{PGL}_2(\mathbb{Z})$ to be the action

$$\begin{array}{ccc} \big(\mathrm{PGL}_{2,q}(\mathbb{Z}) \times \mathbb{Z}_2\big) \times \mathbb{P}^1(\mathbb{Z}(q)) & \longrightarrow & \mathbb{P}^1(\mathbb{Z}(q)) \\ \big((M_q, \epsilon), f\big) & \longmapsto & \begin{cases} M_q \cdot f \text{ if } \epsilon = 1 \\ \overline{M}_q \cdot f \text{ if } \epsilon = -1 \end{cases} \end{array}.$$

Proposition 1.3.15 ([Jou25b])**.** *Let $M \in \mathrm{PGL}_2(\mathbb{Z})$. Let $x \in \mathbb{P}^1(\mathbb{Q})$. Then, if $\det(M) = 1$,*

$$\overline{M}_q \cdot [x]^\sharp = [M \cdot x]^\flat \text{ and } \overline{M}_q \cdot [x]^\flat = [M \cdot x]^\sharp,$$

and if $\det(M) = -1$,

$$\overline{M}_q \cdot [x]^\sharp = [M \cdot x]^\sharp \text{ and } \overline{M}_q \cdot [x]^\flat = [M \cdot x]^\flat.$$

Proof. These equalities are direct consequences of behaviour of $\overline{I}_q$:

$$\overline{I}_q \cdot [x]^\sharp = [x]^\flat \text{ and } \overline{I}_q \cdot [x]^\flat = [x]^\sharp.$$

□

The behavious of $\overline{I}_q$ on q-deformed rational numbers is made explicit below.

Corollary 1.3.16. *For any $x \in \mathbb{P}^1(\mathbb{Q})$,*

$$[x]^\sharp_{q^{-1}} = \frac{q[x]^\flat_q + (1-q)}{(q-1)[x]^\flat_q + 1}, \tag{1.16}$$

$$[x]^\flat_{q^{-1}} = \frac{q[x]^\sharp_q + (1-q)}{(q-1)[x]^\sharp_q + 1}. \tag{1.17}$$

Example 1.3.17. Let us take $M = T$ and $x = n \in \mathbb{N}^*$. We have

$$\begin{aligned}\overline{T}_q \cdot [n]^\sharp = \begin{pmatrix} 1 & q^2 \\ 1-q & q \end{pmatrix} \cdot [n]^\sharp_{q^{-1}} &= \frac{[n]^\sharp_{q^{-1}} + q^2}{(1-q)[n]^\sharp_{q^{-1}} + q} \\ &= q^{n-1}[n]^\sharp_{q^{-1}} + q^{n+1} \\ &= [n]^\sharp_q + q^{n+1} \\ &= [n+1]^\flat.\end{aligned}$$

Corollary 1.3.18 ([LMG21, Jou25b])**.** *Let $x \in \mathbb{P}^1(\mathbb{Q})$. Then*

$$[-x]^\sharp = -q^{-1}[x]^\sharp_{q^{-1}}, \quad \textit{and} \quad \left[\frac{1}{x}\right]^\sharp = \frac{1}{[x]^\sharp_{q^{-1}}},$$

and similarly with the left version $[\cdot]^\flat$.

Remark 1.3.19. The twisted deformation of identity already appeared in [Tho24], where Thomas defined the q-rational transition map $g_q : x \mapsto \frac{1+(x-1)q}{1+(q-1)x}$. Theorem 1.2 of [Tho24] can be understood in light of the following equality

$$\tau \circ g_q = I_q \circ \tau = \overline{I_q}.$$

1.4 Combinatorial aspects of q-rational numbers

This section is devoted to the interpretations of numerators and denominators of q-rational numbers, as generating functions for some combinatorial models. From these interpretations will follow important properties of q-rational numbers, such as positivity or unimodality.

Notation 1.4.1. Let x be a rational number, and $[x]^\square$ its right or left q-deformation. This is a rational fraction in q, so denote by its $A^\square$ (resp. $B^\square$) numerator (resp. its denominator).

$$[x]^\square = \frac{A^\square}{B^\square}.$$

We fix the following conventions.

- $A^\square$ and $B^\square$ are coprime polynomials in q,
- $A^\square$ and $B^\square$ have integer coefficients,
- $B^\square$ has positive dominant coefficient.

The only exception to this convention is for the left deformation of $x = \frac{1}{0}$, where the numerator is 1 and denominator is $1 - q$. We will show that the polynomials $A^\square$ and $B^\square$ have constant sign coefficients, and we will study there degrees.

Notation 1.4.2. We denote by $A \equiv B$ if A and B are two polynomials in q equal up to a power of q or sign.

1.4.1 Quantum matrices and continued fractions

Thanks to the relationship between the modular group and continued fractions, as explained in Definition 1.1.22, one can derive q-analogues of the expansion of a rational number. This is the original definition of right q-rational numbers, in [MGO20]. The formula for left q-rational numbers is written in [BBL23] for the positive case, and in [Ren24] for the negative case.

Corollary 1.4.3 ([MGO20, BBL23, Ren24]). *Let $x \in \mathbb{Q}$ and let $x = [a_1, a_2, \cdots, a_{2m}]$ (resp. $x = [\![c_1, c_2, \cdots, c_k]\!]$) be its even positive (resp. negative) continued fraction expansion. Then*

$$[x]^\sharp = [a_1]^\sharp + \cfrac{q^{a_1}}{[a_2]^\sharp_{q^{-1}} + \cfrac{q^{-a_2}}{\ddots + \cfrac{q^{a_{2m-1}}}{[a_{2m}]^\sharp_{q^{-1}}}}} = [c_1]^\sharp - \cfrac{q^{c_1-1}}{[c_2]^\sharp - \cfrac{q^{c_2-1}}{\ddots - \cfrac{q^{c_{k-1}-1}}{[c_k]^\sharp}}}. \tag{1.18}$$

$$[x]^\flat = [a_1]^\sharp + \cfrac{q^{a_1}}{[a_2]^\sharp_{q^{-1}} + \cfrac{q^{-a_2}}{\ddots + \cfrac{q^{a_{2m-1}}}{[a_{2m}]^\flat_{q^{-1}}}}} = [c_1]^\sharp - \cfrac{q^{c_1-1}}{[c_2]^\sharp - \cfrac{q^{c_2-1}}{\ddots - \cfrac{q^{c_{k-1}-1}}{[c_k]^\flat}}}. \tag{1.19}$$

It is easy to check that the formulas also hold for the odd positive continued fraction expansions. Actually, it holds for any continued fraction expansion of x. This was proven by Leclere and Morier-Genoud (Theorem 4 in [LMG21]) for right q-rational numbers, using equivariance with respect to the action of the modular group. Since left q-rational numbers also have this equivariance property, the argument is the same in the left case.

Example 1.4.4. With $x = \frac{22}{9} = [2, 2, 3, 1]$, one can compute

$$\left[\frac{22}{9}\right]^\sharp = 1 + q + \cfrac{q^2}{1 + q^{-1} + \cfrac{q^{-2}}{1 + q + q^2 + \cfrac{q^3}{1}}} = \frac{1 + 2q + 3q^2 + 4q^3 + 5q^4 + 4q^5 + 2q^6 + q^7}{1 + q + 2q^2 + 2q^3 + 2q^4 + q^5},$$

$$\left[\frac{22}{9}\right]^\flat = 1 + q + \cfrac{q^2}{1 + q^{-1} + \cfrac{q^{-2}}{1 + q + q^2 + \cfrac{q^3}{q^{-1}}}} = \frac{1 + 2q + 3q^2 + 4q^3 + 4q^4 + 3q^5 + 3q^6 + q^7 + q^8}{1 + q + 2q^2 + 2q^3 + q^4 + q^5 + q^6}.$$

All these formulas can be understood in the q-matrices setting, via the q-deformation of matrices $M_+(x)$ and $M_-(x)$ defined in (1.6) and (1.7). In the definitions below, we work in $\mathrm{GL}_2(\mathbb{Z}[q, q^{-1}])$, so we fix representants T_q and S_q as in Definition 1.2.4. Knowing exactly how the powers of q behave will be useful later on.

Definition 1.4.5. Let x be a rational number and let $x = [a_1, a_2, \cdots, a_{2m}]$ be its even positive continued fraction expansion. Define

$$M_{+,q}(x) = (M_+(x))_q = T_q^{a_1} L_q^{a_2} \cdots T_q^{a_{2m-1}} L_q^{a_{2m}}. \tag{1.20}$$

Similarly, let $x = [\![c_1, c_2, \cdots, c_k]\!]$ be the negative continued fraction of x. Define

$$M_{-,q}(x) = (M_-(x))_q = T_q^{c_1} S_q T_q^{c_2} S_q \cdots T_q^{c_k} S_q. \tag{1.21}$$

Proposition 1.4.6 (Proposition 4.3 of [MGO20]). *Let $x \in \mathbb{Q}\backslash\mathbb{Z}_{<0}$ and let $[a_1, a_2, \cdots, a_{2m}]$ be its even positive continued fraction expansion. Let us set $x_{2m-1} = [a_1, a_2, \cdots, a_{2m-1}]$ and*

$$[x]^\sharp = \frac{A^\sharp}{B^\sharp} \text{ and } [x_{2m-1}]^\sharp = \frac{A^\sharp_{2m-1}}{B^\sharp_{2m-1}}.$$

Then it holds

$$M_{+,q}(x) = q^{\min(0,a_1)} \begin{pmatrix} qA^\sharp & A^\sharp_{2m-1} \\ qB^\sharp & B^\sharp_{2m-1} \end{pmatrix}.$$

Likewise, let $x = [\![c_1, c_2, \cdots, c_k]\!]$ be the negative continued fraction expansion of x, and let us set $x_{k-1} = [\![c_1, c_2, \cdots, c_{k-1}]\!]$, with

$$[x_{k-1}]^\sharp = \frac{A^\sharp_{k-1}}{B^\sharp_{k-1}}.$$

Then

$$M_{-,q}(x) = q^{\min(0,c_1-1)+k} \begin{pmatrix} A^\sharp & -q^{c_k-1}A^\sharp_{k-1} \\ B^\sharp & -q^{c_k-1}B^\sharp_{k-1} \end{pmatrix}.$$

Remark 1.4.7. The proposition above does not cover the case of negative integers.
If $x = -n \in \mathbb{Z}_{<0}$, then

$$M_{+,q}(-n) = q^{-n-1} \begin{pmatrix} -q^2[n]^\sharp & -[n+1]^\sharp \\ q^{n+2} & q^{n+1} \end{pmatrix} \text{ and } M_{-,q}(-n) = q^{-n-1} \begin{pmatrix} -q[n]^\sharp & -1 \\ q^{n+1} & 0 \end{pmatrix}.$$

As a byproduct of Proposition 1.4.6, we can compute the degrees of the numerators and denominators in the right deformations of a rational number. The analogue result for the left deformation will be given later, as a consequence of the combinatorial interpretations of q-rationals (Corollary 1.4.65).

Corollary 1.4.8. *Let $x \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$ and let $[x]^\sharp = \frac{A^\sharp}{B^\sharp}$ in reduced form (see Notation 1.4.1). Let $x = [a_1, \cdots, a_{2m}]$ be the even positive continued fraction of x. Then*

$$\deg(A^\sharp) = \max(0, a_1) + a_2 + \cdots + a_{2m} - 1$$
$$\deg(B^\sharp) = \max(0, -a_1) + a_2 + \cdots + a_{2m} - 1.$$

Proof. Let $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ be a matrix whose coefficients are Laurent polynomials in q and suppose that $\deg(C) < \deg(A)$ and $\deg(D) < \deg(B)$. Let $a \in \mathbb{Z}$. We have

$$MT^a_q = \begin{pmatrix} q^aA & [a]_qA + C \\ q^aB & [a]_qB + D \end{pmatrix} \text{ and } ML^a_q = \begin{pmatrix} q^aA + q[a]_qC & C \\ q^aB + q[a]_qD & D \end{pmatrix}.$$

Therefore multiplying either by T^a_q or by L^a_q changes the degree of the polynomials in the first column by $\deg \mapsto \deg + a$, and the polynomials in the second column still have a degree lower than those in the first column.
By Proposition 1.4.6, we then have

$$\deg(A^\sharp) = \sum_{i=1}^{2m} a_i - 1 - \min(0, a_1) = \max(0, a_1) + a_2 + \cdots + a_{2m} - 1.$$

For $B^\sharp$, the formula is the same except that the first step $T^{a_1}_q$ does not contribute, since the bottom-left coefficient of this matrix is 0. □

Remark 1.4.9. Again, the case of nonnegative integers is not covered by Corollary 1.4.8. For $n \in \mathbb{Z}_{>0}$,

$$[-n]_q^\sharp = -q^{-1} - \cdots - q^{-n} = -\frac{[n]_q^\sharp}{q^n},$$

so the numerator has degree $n-1$ and the denominator has degree n.

Motivated by these formulas, we associate an integer $\varepsilon(x)$ to a rational number x.

Definition 1.4.10. Let $x \in \mathbb{Q}$ and let $x = [a_1, \cdots, a_{2m}] = [\![c_1, \cdots, c_k]\!]$ be the even positive and the negative continued fraction of x. We put

$$\varepsilon(x) = \begin{cases} n \text{ if } x = -n \in \mathbb{Z}_{\leq 0}, \\ |a_1| + a_2 + ... + a_{2m} - 1 = |c_1 - 1| + (c_2 - 1) + \cdots + (c_k - 1) \text{ else.} \end{cases}$$

In order to have a similar matrix formula for the left q-rational numbers, we will use odd positive continued fraction expansion, and this will need the extension $\mathrm{PGL}_{2,q}(\mathbb{Z})$ defined in Section 1.3. We use the following elementary product of matrices.

$$T^a J = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{q} T_q^a J_q = \begin{pmatrix} q[a]^\flat & 1 \\ q & 1-q \end{pmatrix}.$$

Remark 1.4.11. Note the useful relation, for any $a_1, a_2 \in \mathbb{Z}$,

$$T_q^{a_1} J_q T_q^{a_2} J_q = t_q T_q^{a_1} L_q^{a_2}.$$

with $t_q = q^2 - q + 1$.

Definition 1.4.12. Let x be a rational number and let $x = [a_1, a_2, \cdots, a_{2m+1}]$ be its odd positive continued fraction expansion. Define

$$M_{+,q}^{\mathrm{odd}}(x) = T_q^{a_1} J_q T_q^{a_2} \cdots T_q^{a_{2m+1}} J_q. \tag{1.22}$$

Proposition 1.4.13 ([Jou25b])**.** *Let x be a rational number and let $[a_1, a_2, \cdots, a_{2m+1}]$ its odd positive continued fraction expansion, with $m \geq 1$. Let us set $x_{2m} = [a_1, \cdots, a_{2m}]$ and*

$$[x]^\flat = \frac{A^\flat}{B^\flat} \text{ and } [x_{2m}]^\flat = \frac{A_{2m}^\flat}{B_{2m}^\flat}.$$

Then it holds

$$M_{+,q}^{\mathrm{odd}}(x) = q^{\min(0,a_1)} t_q^m \begin{pmatrix} qA^\flat & A_{2m}^\flat \\ qB^\flat & B_{2m}^\flat \end{pmatrix}.$$

Proof. We consider $x_{2m-1} = [a_1, \cdots, a_{2m-1}]$, and its right quantization $[x_{2m-1}]^\sharp = A_{2m-1}^\sharp / B_{2m-1}^\sharp$. According to the even case and Remark 1.4.11,

$$T_q^{a_1} J_q T_q^{a_2} J_q \cdots T_q^{a_{2m}} J_q = q^{\min(0,a_1)} t_q^m \begin{pmatrix} qA_{2m}^\sharp & A_{2m-1}^\sharp \\ qB_{2m}^\sharp & B_{2m-1}^\sharp \end{pmatrix}.$$

Therefore

$$M_{+,q}^{\mathrm{odd}}(x) = q^{\min(0,a_1)} t_q^m \begin{pmatrix} q(q[a_{2m+1}]^\flat A_{2m}^\sharp + A_{2m-1}^\sharp) & qA_{2m}^\sharp + (1-q)A_{2m-1}^\sharp \\ q(q[a_{2m+1}]^\flat B_{2m}^\sharp + B_{2m-1}^\sharp) & qB_{2m}^\sharp + (1-q)B_{2m-1}^\sharp \end{pmatrix}.$$

By the continued fraction formula for left q-rationals (1.19)

$$A^\flat = q[a_{2m+1}]^\flat A^\sharp_{2m} + A^\sharp_{2m-1} \text{ and } B^\flat = q[a_{2m+1}]^\flat B^\sharp_{2m} + B^\sharp_{2m-1}.$$

Finally, because $[x_{2m}]^\flat = M_{+,q}(x_{2m}) \cdot \frac{1}{1-q}$, we have $qA^\sharp_{2m} + (1-q)A^\sharp_{2m-1} = A^\flat_{2m}$ and same for denominators. □

Remark 1.4.14. The proposition above does not cover the case of an integer $x = n \in \mathbb{Z}$. In that case, the odd continued fraction expansion is $[n]$, and we have

$$T_q^n J_q = \begin{pmatrix} q^{n+1}[n]^\flat_{q^{-1}} & 1 \\ q & 1-q \end{pmatrix}.$$

Example 1.4.15. Let us detail the example of $x = \frac{7}{5} = [1,2,1,1] = [1,2,2]$. With the even continued fraction, we get

$$T_q L_q^2 T_q L_q = \begin{pmatrix} q^5+2q^4+2q^3+q^2+q & q^3+q^2+q+1 \\ q^4+2q^3+q^2+q & q^2+q+1 \end{pmatrix},$$

so that $\left[\frac{7}{5}\right]^\sharp = \frac{1+q+2q^2+2q^3+q^4}{1+q+2q^2+q3}$. For the odd continued fraction,

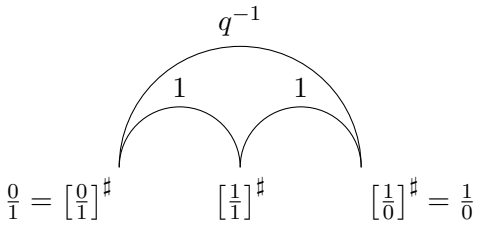
$$\begin{aligned} T_q J_q T_q^2 J_q T_q^2 J_q &= \begin{pmatrix} q^8+2q^6+2q^4+q^3+q & q^5+2q^2-q+1 \\ q^7+q^5+q^4+q^3+q & q^4-q^3+2q^2-q+1 \end{pmatrix} \\ &= t_q \begin{pmatrix} q^6+q^5+2q^4+q^3+q^2+q & q^3+q^2+1 \\ q^5+q^4+q^3+q^2+q & q^2+1 \end{pmatrix}. \end{aligned}$$

thus $\left[\frac{7}{5}\right]^\flat = \frac{1+q+q^2+2q^3+q^4+q^5}{1+q+q^2+q^3+q^4}$.

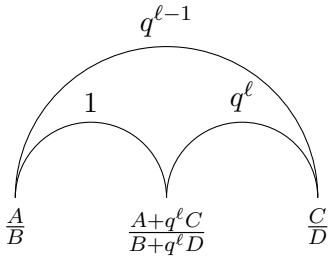
1.4.2 q-Farey graphs

The Farey graph will provide an inductive way of constructing q-rational numbers, by the addition of some weights on edges of the usual Farey graph, and a modified rule taking account of these weights. The resulting picture is called the q-Farey graph. There are actually two versions of q-Farey graph, one for each version (left and right) of q-rationals. The right q-farey graph was constructed by Morier-Genoud and Ovsienko in [MGO20], and the left version is written in [FQ25] and [Ren24].

Definition 1.4.16 ([MGO20])**.** The right q-Farey graph is defined inductively, starting with



and following the weighted rule



It can be shown, by induction, that this construction recovers the right q-rational numbers.

Proposition 1.4.17 ([MGO20])**.** *The vertices of the right q-Farey graph are labelled with the right q-rational numbers.*

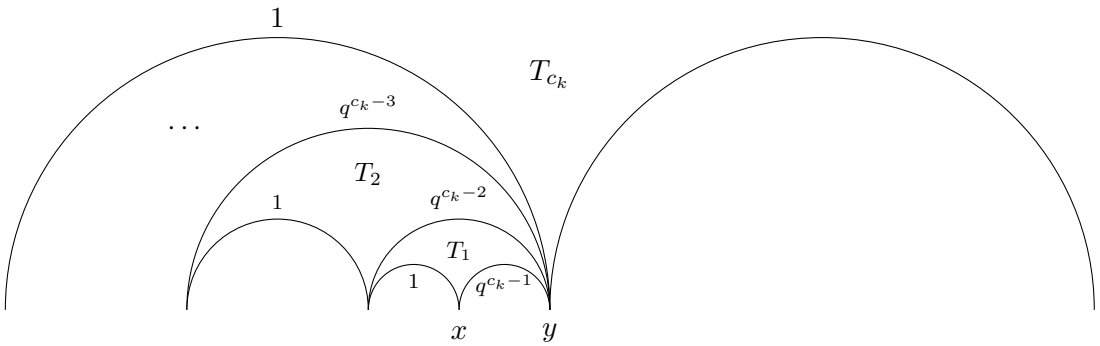
Remark 1.4.18. In the classical Farey graph, two rational numbers are linked if and only if their Farey determinant is 1. In the q-Farey graph, one can check by induction that two q-rational numbers are linked if and only if their Farey determinant is a power of q. The value of the power of q will be computed later using continued fraction expansions.

Lemma 1.4.19. *In the q-Farey graph, the fractions computed with the weighted rule are already reduced.*

Proof. In the initial triangle of the q-Farey, fractions are in reduced form by definition. Consider a triangle which is not the initial one, with vertices $\left(\frac{A}{B}, \frac{A+q^\ell C}{B+q^\ell D}, \frac{C}{D}\right)$, and suppose that the fractions $\frac{A}{B}$ and $\frac{C}{D}$ are reduced. According to previous Remark 1.4.18, there is an integer N such that $B(A+q^\ell C) - A(B+q^\ell D) = q^N$.
Then $\gcd(A+q^\ell C, B+q^\ell D)$ is a power of q. Suppose that q divides $A+q^\ell C$ and $B+q^\ell D$. As $\ell > 0$ (because the triangle is not the initial one), then q divides both A and B, which is impossible. Thus $\gcd(A+q^\ell C, B+q^\ell D) = 1$ and the fraction is reduced. □

Lemma 1.4.20 ([MGO20])**.** *Let x be a positive rational number, and $[\![c_1, \cdots, c_k]\!]$ its negative continued fraction expansion. Then the weight on the edge between x and its right parent in the right q-Farey graph is equal to q^{c_k-1}.*

Proof. Let us denote by y the right parent of x. By Proposition 1.1.16, c_k is the number of triangles incident to y in the subgraph $\mathbb{T}_x$. Denote these triangles $T_1, T_2, \cdots, T_{c_k}$ in decreasing order of depth in the Farey tree. Hence T_1 is the triangle centered at x, T_2 is the triangle centered at the left parent of x, and so on. The vertex y is the right parent in all the triangles T_i, except for the last one T_{c_k} which is centered at y.



According to the q-Farey rule, the edge above triangle T_{c_k-1} is 1, and then it is multiplied by q for each triangle T_i until T_1, so the edge above T_1 has weight q^{c_k-2}, and the edge between x and y has weight q^{c_k-1}. □

Proposition 1.4.21 ([MGO20]). *Let $\frac{a}{b} < \frac{c}{d}$ be a pair of rational numbers with Farey determinant 1. Consider the integers $\varepsilon(\frac{a}{b})$ and $\varepsilon(\frac{c}{d})$ associated to them, as in Definition 1.4.10. Then*

$$B^\sharp C^\sharp - A^\sharp D^\sharp = \begin{cases} q^{\varepsilon(\frac{c}{d})} & \text{if } a < c \\ q^{\varepsilon(\frac{a}{b})} & \text{if } a > c \end{cases}.$$

Morier-Genoud and Ovsienko stated this result only for nonnegative rational numbers, and we extend it here to any rational number.

Proof. Because $\frac{a}{b}$ and $\frac{c}{d}$ have Farey determinant 1, they are linked in the Farey graph, so one of them is the parent of the other.
Case 1 : $\frac{c}{d}$ is the right parent of $\frac{a}{b}$ (which means that $a > c$). Then by Corollary 1.1.18, $[\![c'_1, \cdots, c'_{k'}]\!] = [\![c_1, \cdots, c_{k-1}]\!]$, and by Proposition 1.4.6,

$$M_{-,q}\left(\frac{a}{b}\right) = q^{\min(0,c_1-1)+k} \begin{pmatrix} A^\sharp & -q^{c_k-1}C^\sharp \\ B^\sharp & -q^{c_k-1}D^\sharp \end{pmatrix}.$$

Taking determinant, we get $q^{c_1+\cdots+c_k+k} = q^{2\min(0,c_1-1)+c_k-1+2k}(B^\sharp C^\sharp - A^\sharp D^\sharp)$, and the formula follows.
Case 2 : $\frac{a}{b}$ is the left parent of $\frac{c}{d}$ (which means that $a < c$). Then denote by $\frac{e}{f}$ the right parent of $\frac{c}{d}$, and by $\frac{E^\sharp}{F^\sharp}$ its right q-deformation. By the q-Farey rule (Definition 1.4.16) and the Lemma 1.4.20,

$$C^\sharp = A^\sharp + q^{c'_{k'}-1}E^\sharp \text{ and } D^\sharp = B^\sharp + q^{c'_{k'}-1}F^\sharp.$$

As before, by Proposition 1.4.6,

$$M_{-,q}\left(\frac{c}{d}\right) = q^{\min(0,c'_1-1)+k'} \begin{pmatrix} C^\sharp & -q^{c'_{k'}-1}E^\sharp \\ D^\sharp & -q^{c'_{k'}-1}F^\sharp \end{pmatrix}.$$

Taking determinant, we get $q^{c'_1+\cdots+c'_{k'}+k'} = q^{2\min(0,c'_1-1)+c'_{k'}-1+2k'}(E^\sharp D^\sharp - F^\sharp C^\sharp)$. Using the q-Farey rule, we get

$$q^{c'_1+\cdots+c_{k'}-k'-2\min(0,c'_1-1)} = (D^\sharp(C^\sharp - A^\sharp) - (D^\sharp - B^\sharp)C^\sharp) = B^\sharp C^\sharp - A^\sharp D^\sharp,$$

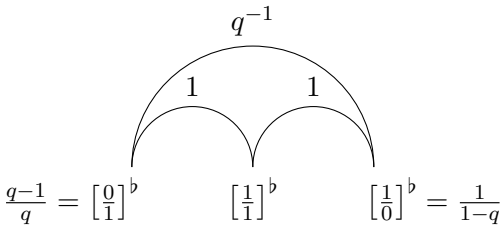
and the formula follows. □

Using the Farey construction of q-rational numbers, one can prove that q-rational numbers satisfy total positivity. It was done by Morier-Genoud and Ovsienko for right q-rational numbers in [MGO20]. In the next section, we will generalize this property to other q-deformations of the Farey determinant.

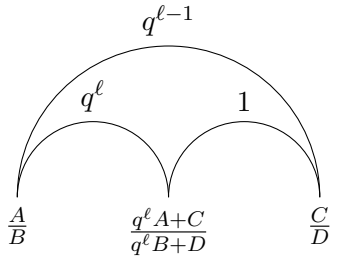
Theorem 1.4.22 (Total positivity [MGO20]). *Let $\frac{a}{b} < \frac{c}{d}$ be two ordered rational numbers. Then the polynomial $B^\sharp C^\sharp - A^\sharp D^\sharp$ has positive integer coefficients.*

Let us now turn toward the left q-rational numbers. Almost everything is symmetric to the right version.

Definition 1.4.23. The left q-Farey graph is defined inductively, starting with



and following the weighted rule



Proposition 1.4.24 ([FQ25, Ren24])**.** *The vertices of the left q-Farey graph are labelled with the left q-rational numbers.*

Remark 1.4.25. Similarly to the right version, in the left q-Farey graph, two rational fractions are linked if and only if their Farey determinant is a power of q. Moreover, the left weighted rule produces fractions in reduced form.

The following lemma is the left version of Lemma 1.4.20, we state it without proof as the arguments are the same.

Lemma 1.4.26. *Let x be a positive rational number, and $[a_1, \cdots, a_{2m}]$ its even positive continued fraction expansion. Then the weight on the edge between x and its left parent in the left q-Farey graph is equal to $q^{a_{2m}}$, unless $x = \frac{1}{n}$, $n \in \mathbb{N}^*$, in which case the weight is q^{n-1}.*

Before considering the total positivity for left q-rational numbers, we state the simple positivity.

Proposition 1.4.27. *Let $\frac{a}{b}$ be a positive rational number. Then its left q-deformation $\frac{A^\flat}{B^\flat}$ is a quotient of polynomials with nonnegative integer coefficients : $A^\flat$, $B^\flat \in \mathbb{N}[q]$.*

Proof. By the symmetry $\left[\frac{1}{x}\right]_q^\flat = \frac{1}{[x]_{q^{-1}}^\flat}$, we can suppose without loss of generality that $\frac{a}{b}$ is greater than 1.
Let us suppose first that the right parent of $\frac{a}{b}$ is $\frac{1}{0}$. Then $\frac{a}{b}$ is an integer, and its left q-deformation has nonnegative coefficients (see (1.11) above).
Let us now suppose that the right parent of $\frac{a}{b}$ is not $\frac{1}{0}$. By induction, suppose the two parents of $\left[\frac{a}{b}\right]^\flat$ has nonnegative coefficients. Then by the left q-Farey rule in Definition 1.4.23, $A^\flat$ and $B^\flat$ also have nonnegative coefficients. □

1.4.3 q-Farey determinants and positivity

We define four versions of q-deformed Farey determinant, according to the choice of left or right q-deformation for each rational of a pair. This section is part of a collaborative work with Olga Paris-Romaskevich and Alexander Thomas [JPRT26].

Fix a pair of rational numbers $(\frac{a}{b}, \frac{c}{d})$ and denote their q-deformations by

$$\left[\frac{a}{b}\right]^\sharp = \frac{A^\sharp}{B^\sharp}\ ,\ \left[\frac{a}{b}\right]^\flat = \frac{A^\flat}{B^\flat}\ ,\ \left[\frac{c}{d}\right]^\sharp = \frac{C^\sharp}{D^\sharp}\ ,\ \left[\frac{c}{d}\right]^\flat = \frac{C^\flat}{D^\flat}.$$

As before, the convention is that in a fraction $\frac{R}{S}$, R and S are coprime polynomials in q with integer coefficients, and S has positive dominant coefficient. The only exception is for $\left[\frac{1}{0}\right]^\flat = \frac{1}{1-q}$.

Definition 1.4.28. The *q-deformed Farey determinants* of the pair $(\frac{a}{b}, \frac{c}{d})$ are

$$d_F^{\sharp\sharp}\left(\tfrac{a}{b}, \tfrac{c}{d}\right) = B^\sharp C^\sharp - A^\sharp D^\sharp \quad , \quad d_F^{\sharp\flat}\left(\tfrac{a}{b}, \tfrac{c}{d}\right) = B^\sharp C^\flat - A^\sharp D^\flat,$$

$$d_F^{\flat\sharp}\left(\tfrac{a}{b}, \tfrac{c}{d}\right) = B^\flat C^\sharp - A^\flat D^\sharp \quad , \quad d_F^{\flat\flat}\left(\tfrac{a}{b}, \tfrac{c}{d}\right) = \frac{B^\flat C^\flat - A^\flat D^\flat}{q^2-q+1}.$$

Notice that we divide by $t_q = q^2 - q + 1$ in the definition of the left-left q-Farey determinant $d_F^{\flat\flat}$. This will be explained, and for this we need the following lemma studying special values of q-rationals at 6-th roots of unity. These properties complete previous works on special values: [MGO20, Section 1.4] for $q = -1$, [KMR+25, Section 7] for 3rd roots of unity and [Lec24, BRY26] for roots of unity of order at most 5.

Lemma 1.4.29. *Denote by $\sigma := \frac{1+i\sqrt{3}}{2}$ and by $\mathbb{U}_6$ the set of 6-th roots of unity. Then*

1. $B^\sharp(\sigma) + (\sigma - 1)A^\sharp(\sigma) \in \mathbb{U}_6$,
2. $B^\flat(\sigma) + (\sigma - 1)A^\flat(\sigma) = 0$,
3. $A^\flat(\sigma)$ *and* $B^\flat(\sigma)$ *are in* $\mathbb{U}_6$.

Proof. All the identities hold for $\frac{a}{b} = 0$. It is then sufficient to show that the properties are invariant under T and S, since the modular group acts transitively on $\mathbb{P}^1(\mathbb{Q})$.
For $\square \in \{\sharp, \flat\}$, the expression $B^\square(q) + (q-1)A^\square(q)$ transforms under T into

$$B^\square(q) + (q-1)(qA^\square + B^\square) = q(B^\square(q) + (q-1)A^\square(q))$$

up to a power of q. Hence, evaluating at $q = \sigma$ does not change the fact of being zero or being in $\mathbb{U}_6$.

Similarly, the transformation under S gives $qA^\square(q) - (q-1)B^\square(q)$, up to sign. Evaluating at $q = \sigma$ and using $\sigma^2 - \sigma + 1 = 0$, this gives $-\sigma^2(B^\square(\sigma) + (\sigma-1)A^\square(\sigma))$. Again, this preserves the fact of being zero or in $\mathbb{U}_6$. This finishes the proof of (1) and (2).
For (3), note that $A^\flat(\sigma)$ transforms under T into

$$\sigma A^\flat(\sigma) + B^\flat(\sigma) = A^\flat(\sigma)$$

using the identity from (2). Under S it transforms into $\pm B^\flat(\sigma) = \mp\sigma^2 A^\flat(\sigma)$, where we used again (2). Therefore, $A^\flat(\sigma) \in \mathbb{U}_6$. Finally, since $B^\flat(\sigma) = -\sigma^2 A^\flat(\sigma)$, we also get $B^\flat(\sigma) \in \mathbb{U}_6$. □

The classical Farey determinant is invariant under the modular action. We will prove a q-version of this fact.

Proposition 1.4.30 ([JPRT26]). *The Farey determinant is invariant under the* $\mathrm{PSL}_2(\mathbb{Z})$*-action. All pairs of Farey determinant* n *belong to the same* $\mathrm{PSL}_2(\mathbb{Z})$*-orbit, that of* $\left(\frac{1}{0}, \frac{k}{n}\right)$ *with some* $k \in [\![0, n-1]\!]$.

Proof. The invariance of the Farey determinant is clear from the matrix perspective: the Farey determinant is the (absolute value) of the determinant of the matrix $M = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. Since the $\mathrm{PSL}_2(\mathbb{Z})$-action preserves the space of reduced fractions, its action on pairs is simply by matrix multiplication. Hence the Farey determinant is unchanged.

Since $\mathrm{PSL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$, we can send one element of a pair to $\frac{1}{0}$. The other element is then of the form $\frac{k}{n}$, where n is the Farey determinant of the original pair. Using the action of $T \in \mathrm{Stab}(\frac{1}{0})$, we can impose $k \in [\![0, n-1]\!]$. □

Proposition 1.4.31 ([JPRT26]). *The four* q*-Farey determinants are invariant under the* $\mathrm{PSL}_2(\mathbb{Z})$*-action, up to powers of* q *and sign.*

Proof. It is sufficient to compute the change of $d_F^{\triangle\square}$ under T and S (modulo some power of q):

$$A^{\square}D^{\triangle} - B^{\square}C^{\triangle} \overset{T}{\longmapsto} (qA^{\square} + B^{\square})D^{\triangle} - B^{\square}(qC^{\triangle} + D^{\triangle}) = q(A^{\square}D^{\triangle} - B^{\square}C^{\triangle})$$
$$A^{\square}D^{\triangle} - B^{\square}C^{\triangle} \overset{S}{\longmapsto} \pm(qB^{\square}C^{\triangle} - qA^{\square}D^{\triangle}) = \pm q(A^{\square}D^{\triangle} - B^{\square}C^{\triangle}).$$

□

Recall that we denote by $A \equiv B$ if A and B are two polynomials in q equal up to a power of q or sign.

The four q-Farey determinants behave nicely under the duality $q \mapsto q^{-1}$, showing that there are essentially only two q-Farey determinants.

Proposition 1.4.32 ([JPRT26]). *For any two rational numbers,* $d_F^{\flat\flat}$ *is a polynomial and we have*

$$d_F^{\flat\sharp}(q) \equiv d_F^{\sharp\flat}(q^{-1}),$$
$$d_F^{\flat\flat}(q) \equiv d_F^{\sharp\sharp}(q^{-1}).$$

Proof. We can use the $\mathrm{PSL}_2(\mathbb{Z})$-symmetry from Proposition 1.4.31 to send the first rational to $\frac{1}{0}$. The second one is then sent to some rational $\frac{r}{s}$.
Since $\left[\frac{1}{0}\right]^{\sharp} = \frac{1}{0}$ and $\left[\frac{1}{0}\right]^{\flat} = \frac{1}{1-q}$, to prove the first identity we compare $d_F^{\sharp\flat}(q^{-1}) = S^{\flat}(q^{-1})$ to $d_F^{\flat\sharp}(q) = S^{\sharp}(q) - (1-q)R^{\sharp}(q)$.
By the twisted formula (1.17),

$$\frac{R^{\flat}(q^{-1})}{S^{\flat}(q^{-1})} = \left[\frac{r}{s}\right]_{q^{-1}}^{\flat} = \frac{(1-q)S^{\sharp}(q) + qR^{\sharp}(q)}{S^{\sharp}(q) + (q-1)R^{\sharp}(q)}. \tag{1.23}$$

In order to identify the denominators of the two sides, we have to check that the fraction on the right hand side is reduced:

$$\begin{aligned}\gcd((1-q)S^\sharp + qR^\sharp, S^\sharp + (q-1)R^\sharp) &= \gcd((q^2-q+1)R^\sharp, S^\sharp + (q-1)R^\sharp)\\ &= \gcd(R^\sharp, S^\sharp + (q-1)R^\sharp)\\ &= 1,\end{aligned}$$

where we used an elementary operation in the first line, then by point (1) of Lemma 1.4.29, the fact that $\gcd(q^2-q+1, S^\sharp + (q-1)R^\sharp) = 1$ (σ is a root of q^2-q+1), and finally that $\gcd(R^\sharp, S^\sharp) = 1$. This proves

$$d_F^{\flat\sharp}(q) = S^\sharp(q) + (q-1)R^\sharp(q) = q^\alpha S^\flat(q^{-1}) \equiv d_F^{\sharp\flat}(q^{-1}).$$

For the second case, we notice that the polynomial $S^\flat(q) + (q-1)R^\flat(q)$ evaluates to 0 at $q = \sigma$, by Lemma 1.4.29 point (2), hence is divisible by $q^2 - q + 1 = (q-\sigma)(q-\bar\sigma)$. We then have to compare $d_F^{\sharp\sharp}(q^{-1}) = S^\sharp(q^{-1})$ to $(q^2-q+1)d_F^{\flat\flat}(q) = S^\flat(q) + (q-1)R^\flat(q)$. We use the twisted action again (1.16) to get

$$\frac{R^\sharp(q^{-1})}{S^\sharp(q^{-1})} = \left[\frac{r}{s}\right]^\sharp_{q^{-1}} = \frac{(1-q)S^\flat(q) + qR^\flat(q)}{S^\flat(q) + (q-1)R^\flat(q)}. \tag{1.24}$$

The fraction on the right hand side is not reduced this time since

$$\begin{aligned}\gcd((1-q)S^\flat + qR^\flat, S^\flat + (q-1)R^\flat) &= \gcd((q^2-q+1)R^\flat, S^\flat + (q-1)R^\flat)\\ &= (q^2-q+1)\gcd\left(R^\flat, \frac{S^\flat + (q-1)R^\flat}{q^2-q+1}\right)\\ &= (q^2-q+1)\gcd(R^\flat, S^\flat + (q-1)R^\flat)\\ &= q^2-q+1,\end{aligned}$$

where we used point (2) of Lemma 1.4.29 in the second line. For the third line, point (3) of Lemma 1.4.29 implies $\gcd(R^\flat, q^2-q+1) = 1$. Therefore,

$$d_F^{\sharp\sharp}(q^{-1}) = S^\sharp(q^{-1}) \equiv \frac{S^\flat(q) + (q-1)R^\flat(q)}{q^2-q+1} = d_F^{\flat\flat}(q).$$

□

As a consequence, we get a positivity property of all q-Farey determinants. We use the notation $P > 0$ if the polynomial P has positive coefficients.

Theorem 1.4.33 ([JPRT26]). *If $\frac{a}{b} < \frac{c}{d}$, then the four q-Farey determinants have positive integer coefficients.*

$$\begin{aligned} B^\sharp C^\sharp - A^\sharp D^\sharp > 0 \quad &, \quad B^\sharp C^\flat - A^\sharp D^\flat > 0\,,\\ B^\flat C^\sharp - A^\flat D^\sharp > 0 \quad &, \quad \frac{B^\flat C^\flat - A^\flat D^\flat}{q^2-q+1} > 0\,.\end{aligned}$$

Proof. We show that for any pair of distinct rationals, the q-Farey determinants have constant sign coefficients (this property is invariant under T and S). The proposition then

follows, since the sign can be determined by evaluating at $q = 1$.

We have already seen that the Farey determinants do not change under T and S (modulo a power of q). We can send $\frac{a}{b}$ to $\frac{0}{1}$, and then $d_F^{\sharp\square} = C^{\square}$, which has positive coefficients since $0 < \frac{c}{d}$. This proves the proposition for $d_F^{\sharp\sharp}$ and $d_F^{\sharp\flat}$. The remaining two cases follow from the relations between the q-Farey determinants given in Proposition 1.4.32. □

As a consequence, we recover the well-orderedness of q-rationals, mentionned in [BBL23].

Corollary 1.4.34. *For $\frac{a}{b} < \frac{c}{d}$ and $q \in (0,1)$, we have*

$$\left[\frac{a}{b}\right]_q^{\flat} < \left[\frac{a}{b}\right]_q^{\sharp} < \left[\frac{c}{d}\right]_q^{\flat} < \left[\frac{c}{d}\right]_q^{\sharp}.$$

Another consequence arises by combining Proposition 1.4.32 with results about palindromic denominators in q-rationals from [KMR+25, RY25]. We will need this result in Chapter 3.

Corollary 1.4.35. *Consider a rational number $\frac{k}{n}$ in reduced form.*
(i) If n divides k^2+1, then $N^{\sharp} + (q-1)K^{\sharp} \equiv N^{\flat}$.
(ii) If n divides k^2-1, then $N^{\flat} + (q-1)K^{\flat} \equiv (q^2-q+1)N^{\sharp}$.

Proof. From [RY25, Theorem 1.2], we know that $n \mid k^2+1$ implies that $N^{\flat}$ is palindromic. Hence by the action of the twisted identity $\overline{I}_q$ (see denominator of Equation (1.17)), we see that

$$N^{\flat}(q) = q^{\deg(N^{\flat})} N^{\flat}(q^{-1}) \equiv (N^{\sharp} + (q-1)K^{\sharp}).$$

Similarly, from [KMR+25, Corollary 3.8], we know that $n \mid k^2-1$ implies that $N^{\sharp}$ is palindromic. Hence the denominator of Equation (1.16) concludes. □

1.4.4 Triangulated polygons, fence posets and snake graphs

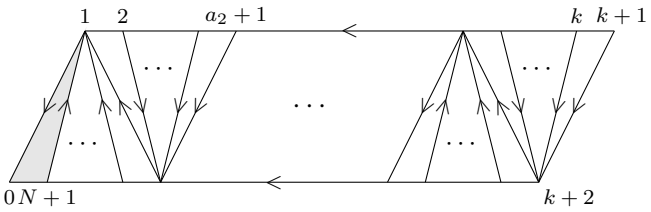
We now present three equivalent combinatorial models described by q-rational numbers. Throughout this subsection, we consider a positive rational number x, with continued fraction expansions $x = [a_1, a_2, \cdots, a_{2m}] = [\![c_1, c_2, \cdots, c_k]\!]$.

Denote $N = \sum_{i=1}^{2m} a_i = \sum_{j=1}^{k} (c_j - 1) + 1 = \varepsilon(x) + 1$.

Definition 1.4.36. Let $\mathbb{T}_x$ be the triangulated polygon associated to x, in the sense of Definition 1.1.5. The *oriented triangulated polygon* $\overrightarrow{\mathbb{T}}_x$ is obtained by endowing the edges of $\mathbb{T}_x$ with an orientation, such that

- all horizontal edges are oriented from right to left,
- in any down-based triangle, the right-most edge is oriented upward,
- and in any up-based triangle, the right-most edge is oriented downward,
- the left-most edge has the opposite orientation of the other interior edge sharing the same triangle.

The polygon $\overrightarrow{\mathbb{T}}_x$ has $N+2$ vertices, and we label them 0, 1, ..., $N+1$, turning in clokwise order starting at the left-most vertex. Note that this left-most vertex is down if $x > 1$ and up if $x \leq 1$.

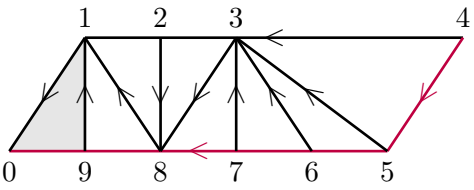
$\overrightarrow{\mathbb{T}}_x =$ 

The first triangle of $\overrightarrow{\mathbb{T}}_x$ (defined by vertices 0, 1 and $N+1$) plays a specific role, we will denote it by Δ_0 and color it in gray.

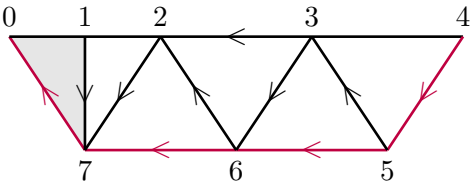
We consider one special path in this graph $\overrightarrow{\mathbb{T}}_x$.

Definition 1.4.37. The *bottom path* π_{bottom} in $\overrightarrow{\mathbb{T}}_x$ is the path going from the right-most vertex to vertex 0, passing by vertices $k+1$, $k+2$,..., $N+1$.
In general, a path in $\overrightarrow{\mathbb{T}}_x$ is a sequence of edges $e_1, e_2, \cdots, e_\ell$ such that each edge e_i is oriented according to the orientation of $\overrightarrow{\mathbb{T}}_x$ and the target of e_i is the starting point of e_{i+1}.

Example 1.4.38. Take $x = \frac{22}{9} = [2,2,3,1] = [\![3,2,5]\!]$. The associated triangulated polygon and the bottom path are depicted below.



Example 1.4.39. Let us make an example with a rational number $0 < x < 1$. Take $x = \frac{5}{13} = [0,2,1,1,1,1] = [\![1,2,3,3]\!]$. The oriented triangulated polygon is then as follows.



Definition 1.4.40. Given a path π in the oriented triangulated polygon $\overrightarrow{\mathbb{T}}_x$, its *area* $\mathrm{ar}(\pi)$ is the number of triangles enclosed between π and the bottom path π_{bottom}. For such a triangle Δ, we write $\Delta \in \pi$.

Theorem 1.4.41 ([LMG23])**.** *Let* $[x]^\sharp = \frac{A^\sharp}{B^\sharp}$ *be the right q-deformation of* x*, in reduced form. Then*

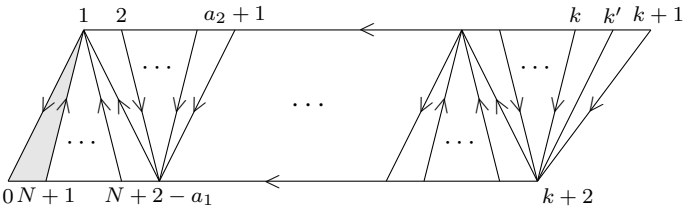
$$qA^\sharp(q) = \sum_{\substack{\pi: k+1 \to 0 \\ \Delta_0 \in \pi}} q^{\mathrm{ar}(\pi)} \text{ and } B^\sharp(q) = \sum_{\substack{\pi: k+1 \to 0 \\ \Delta_0 \notin \pi}} q^{\mathrm{ar}(\pi)}.$$

Remark 1.4.42. Our presentation is slightly different from the one used in [LMG23]. Indeed, we have chosen the setting developed by Aval, Labbé [AL25] which covers all positive rational numbers instead of only the rationals greater than 1.

There is an analogous theorem for left q-rational numbers, but we need to split the last triangle in two.

Definition 1.4.43. Let $\overrightarrow{\mathbb{T}}^{\flat}_x$ be the oriented triangulated polygon obtained from $\overrightarrow{\mathbb{T}}_x$ by cutting in two triangles the right-most triangle, with a non-oriented edge.

$\overrightarrow{\mathbb{T}}^{\flat}_x =$

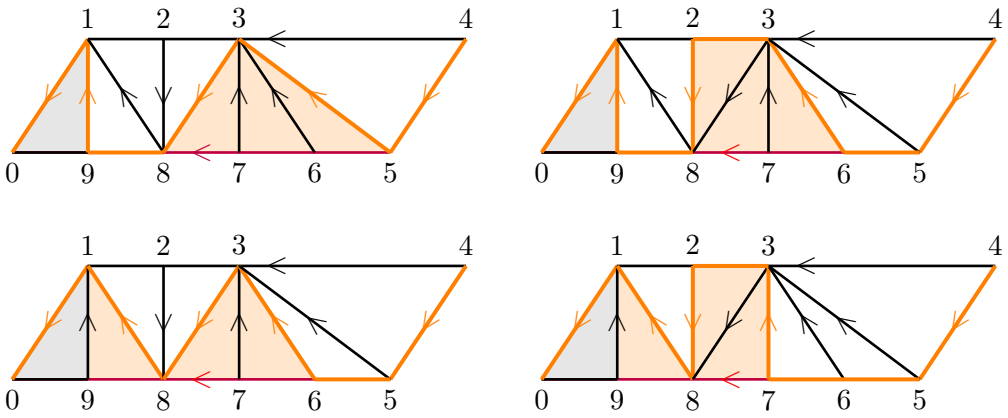


Note that the additional edge is not oriented. This means that paths in $\overrightarrow{\mathbb{T}}^{\flat}_x$ are not allowed to go along this edge. The definition of the bottom path in $\overrightarrow{\mathbb{T}}^{\flat}_x$ remains unchanged, as the area of a given path. The result below is a translation of the combinatorial model stated in Appendix of [BBL23].

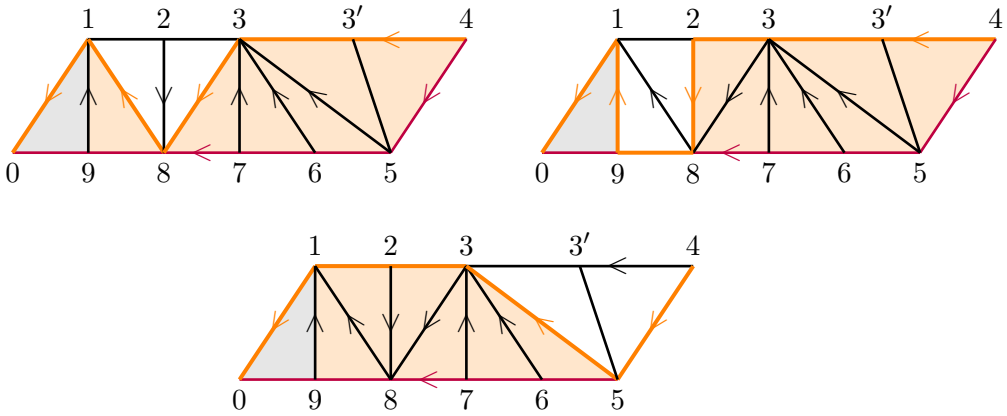
Theorem 1.4.44 ([BBL23]). *Let $[x]^{\flat} = \frac{A^{\flat}}{B^{\flat}}$ be the left q-deformation of x, in reduced form. Then*

$$qA^{\flat}(q) = \sum_{\substack{\pi: k+1 \to 0 \\ \Delta_0 \in \pi}} q^{\mathrm{ar}(\pi)} \text{ and } B^{\flat}(q) = \sum_{\substack{\pi: k+1 \to 0 \\ \Delta_0 \notin \pi}} q^{\mathrm{ar}(\pi)}.$$

Example 1.4.45. Consider as before $x = \frac{22}{9}$, and enumerate all the paths containing Δ_0 of area $4 = 3+1$ in $\overrightarrow{\mathbb{T}}_x$. There are four of them,



This matches with $\left[\frac{22}{9}\right]^{\sharp} = \frac{1+2q+3q^2+4q^3+5q^4+4q^5+2q^6+q^7}{1+q+2q^2+2q^3+2q^4+q^5}$.
In the left version, there are three paths containing Δ_0 of area $7 = 6+1$, pictured below.
This matches with $\left[\frac{22}{9}\right]^{\flat} = \frac{1+2q+3q^2+4q^3+4q^4+3q^5+3q^6+q^7+q^8}{1+q+2q^2+2q^3+q^4+q^5+q^6}$.

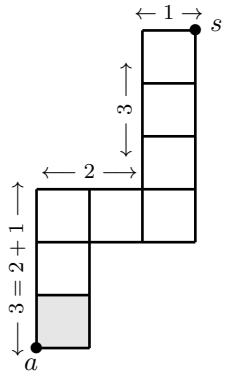


There is a one-to-one correspondence between the combinatorics of paths in triangulated polygons and the combinatorics of paths in snake graphs. These are bipartite graphs obatined by the concatenation of square tiles. They were used by Propp to study the combinatorics of type A cluster algebras [Pro20]. Indeed, snake graphs are a key tool in the understanding of cluster algebra combinatorics, and they were extensively studied for this purpose [MSW11, CS13, Rab18, Cla20]. In Chapter 2, we will explain the construction of dual snake graphs, and the notion of perfect matchings, central in the theory of cluster algebras. In other contexts, snake graphs are viewed as boundaries of Young tableaux, see for example [Sta11a, p.345].

Definition 1.4.46. A *tile* is a square of fixed size in the plane whose sides are parallel or perpendicular to the canonical basis. A *snake graph* is then a concatenation of tiles G_1, G_2,..., G_ℓ such that for each $i \in [\![1, \ell - 1]\!]$, the tiles G_i and G_{i+1} share exactly one side, which must be the upper side or the right side.

Definition 1.4.47. The *snake graph associated to* x is the snake graph whose upper boundary has $a_1 + 1$ edges going up, then a_2 edges going right, then a_3 edges going up, and so on. It is denoted by $\mathcal{G}_x$, and has N tiles. Denote by a the bottom left corner of $\mathcal{G}_x$ and s its top right corner.
The bottom left tile of $\mathcal{G}_x$ plays a specific role, we denote it by G_0 and color it in gray.

Example 1.4.48. We continue our example $x = \frac{22}{9} = [2, 2, 3, 1]$. The snake graph associated to it is

$\mathcal{G}_x =$ 

Definition 1.4.49. A *lattice path* in the snake graph $\mathcal{G}_x$ is a path along the edges of $\mathcal{G}_x$, going from s to a, whose steps are only down or left.

Given such a lattice path π, its *area* $\mathrm{ar}(\pi)$ is the number of tiles enclosed between the path and the lower boundary of $\mathcal{G}_x$.

Proposition 1.4.50. *There are area preserving bijections*

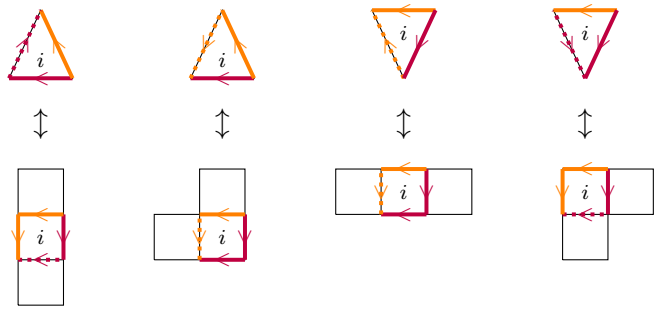
$$\left\{ \text{paths } \pi : k+1 \to 0 \text{ in } \overrightarrow{\mathbb{T}}_x \text{ s.t. } \Delta_0 \in \pi \right\} \longleftrightarrow \{ \text{lattice paths } s \to a \text{ in } \mathcal{G}_x \text{ above } G_0 \}.$$

$$\left\{ \text{paths } \pi : k+1 \to 0 \text{ in } \overrightarrow{\mathbb{T}}_x \text{ s.t. } \Delta_0 \notin \pi \right\} \longleftrightarrow \{ \text{lattice paths } s \to a \text{ in } \mathcal{G}_x \text{ below } G_0 \}.$$

Proof. The snake graph $\mathcal{G}_x$ is equivalently obtained by reading the polygon $\overrightarrow{\mathbb{T}}_x$ from left to right, starting with the second triangle, and replacing each triangle by a tile, such that

- if the ith triangle has a downward right edge, then the tile G_{i+1} is on the right of tile G_i,
- if the ith triangle has an upward right edge, then the tile G_{i+1} is above the tile G_i.

Hence each triangle in $\overrightarrow{\mathbb{T}}_x$ corresponds to a tile in $\mathcal{G}_x$, and a path π in $\overrightarrow{\mathbb{T}}_x$ corresponds to a path in $\mathcal{G}_x$, according to the rules illustrated below.

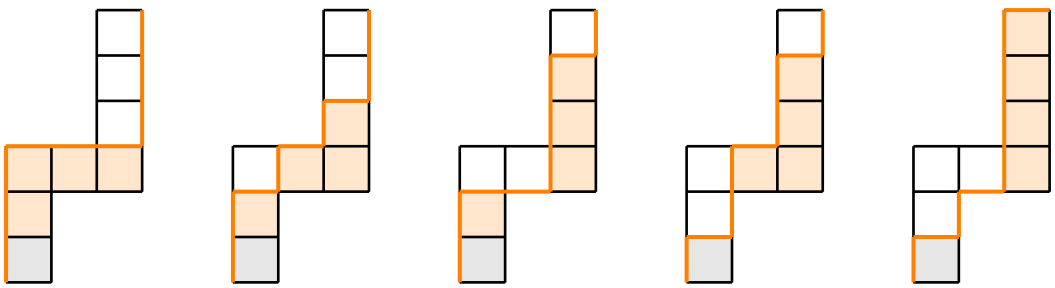


These rules also preserve area of the path, and give the bijection of the proposition. □

Corollary 1.4.51. *Let* $[x]^\sharp = \frac{A^\sharp}{B^\sharp}$ *be the right* q*-deformation of* x*. Then*

$$qA^\sharp(q) = \sum_{\substack{\pi : s \to a \text{ in } \mathcal{G}_x \\ \text{above } G_0}} q^{\mathrm{ar}(\pi)} \text{ and } B^\sharp(q) = \sum_{\substack{\pi : s \to a \text{ in } \mathcal{G}_x \\ \text{below } G_0}} q^{\mathrm{ar}(\pi)}.$$

Example 1.4.52. Keeping our example $x = \frac{22}{9}$, we enumerate five paths in $\mathcal{G}_x$ that contain G_0 and of area $5 = 4 + 1$.



This matches with $\left[\frac{22}{9}\right]^\sharp = \frac{1+2q+3q^2+4q^3+5q^4+4q^5+2q^6+q^7}{1+q+2q^2+2q^3+2q^4+q^5}$.

The left q-deformation also has a snake graph interpretation.

Definition 1.4.53. The *left area* of a lattice path π in $\mathcal{G}_x$, denoted by $\mathrm{ar}^\flat(\pi)$, is the number of tiles enclosed between π and the lower boundary of $\mathcal{G}_x$, the top-right tile counting twice.

Corollary 1.4.54. *Let* $[x]^\flat = \frac{A^\flat}{B^\flat}$ *be the left* q*-deformation of* x*. Then*

$$qA^\flat(q) = \sum_{\substack{\pi: s \to a \text{ in } \mathcal{G}_x \\ \text{above } G_0}} q^{\mathrm{ar}^\flat(\pi)} \text{ and } B^\flat(q) = \sum_{\substack{\pi: s \to a \text{ in } \mathcal{G}_x, \\ \text{below } G_0}} q^{\mathrm{ar}^\flat(\pi)}.$$

Example 1.4.55. Among the five paths of area 5 computed in the previous example, for $x = \frac{22}{9}$, the last one has left area 6 because it encloses the top-right tile which counts twice. Thus the coefficient of q^4 in the numerator of $\left[\frac{22}{9}\right]^\flat$ is equal to 4.

From this snake graph model, we derive a third equivalent setting based on fence posets. This point of view was used by Oğuz and Ravichandran to prove the unimodalilty conjecture formulated in [MGO20].

Definition 1.4.56. A *fence poset* is a partially ordered set whose Hasse diagram is linear. The *fence poset associated to* x, denoted by $\mathcal{F}_x$, has N elements $v_0, v_1, v_2, \cdots, v_{N-1}$ and cover relations

$$v_0 \lhd v_1 \lhd v_2 \lhd \cdots \lhd v_{a_1} \rhd v_{a_1+1} \rhd \cdots \rhd v_{a_1+a_2} \lhd v_{a_1+a_2+1} \lhd \cdots \lhd v_{N-a_{2m}} \rhd \cdots \rhd v_{N-1}.$$

The *left fence poset* $\mathcal{F}_x^\flat$ associated to x is the one obtained from $\mathcal{F}_x$ by adding a last element v_N such that $v_{N-1} < v_N$.
The first vertex v_0 plays a specific role.

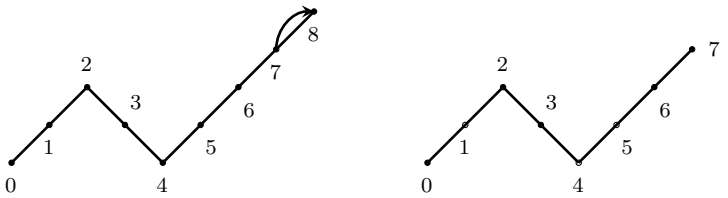
Definition 1.4.57. Let $\mathcal{P}$ be a poset. An *ordered ideal* I of $\mathcal{P}$ is a subset $I \subset \mathcal{P}$ such that for any $x < y$, if $y \in I$ then $x \in I$.

We define a modified version of ordered ideals that will be useful for the left version of q-rationals.

Definition 1.4.58. Let $\mathcal{F}$ be a fence poset, with last two vertices v_{N-1} and v_N. An *admissible ordered ideal* I of $\mathcal{F}$ is an ideal $I \subset \mathcal{F}$ such that $v_N \in I$ if and only if $v_{N-1} \in I$.

To describe this condition visually, we often represent the fence poset with an arrow going from v_{N-1} to v_N.

Example 1.4.59. The fence poset $\mathcal{F}_{\frac{22}{9}}^\flat$ (on the left) and $\mathcal{F}_{\frac{22}{9}}$ (on the right) are depicted below.



Remark 1.4.60. The set of all ordered ideals of a poset $\mathcal{P}$ is a distributive lattice, ranked by the cardinal of ordered ideals. See [Sta11b].

Proposition 1.4.61. *Number the tiles of $\mathcal{G}_x$ from bottom to top G_0, G_1, G_2,... ,G_{N-1}. Then there is a well defined map*

$$\begin{array}{rccc} \phi: & \{ \text{ lattice paths } s \to a \text{ in } \mathcal{G}_x\} & \longrightarrow & \{ \text{ ordered ideals in } \mathcal{F}_x\} \\ & \pi & \longmapsto & \{v_i \mid \pi \text{ is over the tile } G_i\}. \end{array}$$

This map is a bijection, it satisfies $|\phi(\pi)| = \mathrm{ar}(\pi)$, and $\phi(\pi)$ contains v_0 if and only if π is above the tile G_0.

Proof. Note first that replacing each tile of $\mathcal{G}_x$ by a vertex and adding an edge between adjacent tiles, we get the Hasse diagram of the fence poset $\mathcal{F}_x$, up to a slight rotation. Thus a vertex v_i is lower than v_j in the poset $\mathcal{F}_x$ if and only if the tile G_i is either below or on the right of the tile G_j.
Let π be a lattice path from s to a in $\mathcal{G}_x$. By definition of lattice path, if a tile G_j is below π, then any tile below G_j or on the right of it is also under π. This shows that the map ϕ is well-defined.
The following map is the inverse of ϕ, thus it is a bijection.

$$\begin{array}{ccc} \{ \text{ ordered ideals in } \mathcal{F}_x\} & \longrightarrow & \{ \text{ lattice paths } s \to a \text{ in } \mathcal{G}_x\} \\ I & \longmapsto & \text{the path going above } G_i \text{ only for } i \in I \end{array}.$$

This also proves the equality $|\phi(\pi)| = \mathrm{ar}(\pi)$. □

Remark 1.4.62. This endows the set of lattice paths in $\mathcal{G}_x$ and the set of paths in $\overrightarrow{\mathbb{T}}_x$ with a distributive lattice structure.

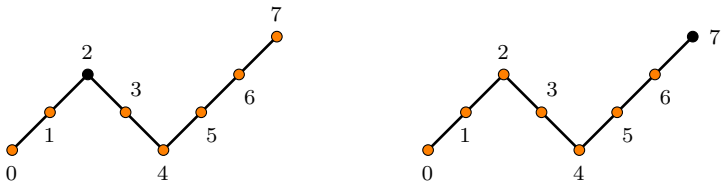
Corollary 1.4.63. *Let $[x]^\sharp = \frac{A^\sharp}{B^\sharp}$ be the right q-deformation of x. Then*

$$qA^\sharp(q) = \sum_{\substack{I \text{ ordered ideal of } \mathcal{F}_x \\ v_0 \in I}} q^{|I|} \text{ and } B^\sharp(q) = \sum_{\substack{I \text{ ordered ideal of } \mathcal{F}_x \\ v_0 \notin I}} q^{|I|}.$$

Let $[x]^\flat = \frac{A^\flat}{B^\flat}$ be the left q-deformation of x. Then

$$qA^\flat(q) = \sum_{\substack{I \text{ admissible ideal of } \mathcal{F}_x^\flat \\ v_0 \in I}} q^{|I|} \text{ and } B^\flat(q) = \sum_{\substack{I \text{ admissible ideal of } \mathcal{F}_x^\flat \\ v_0 \notin I}} q^{|I|}.$$

Example 1.4.64. Always with $x = \frac{22}{9}$, we enumerate two ordered ideals of size $7 = 6+1$ containing v_0 in $\mathcal{F}_{\frac{22}{9}}$.



This matches with $\left[\frac{22}{9}\right]^\sharp = \frac{1+2q+3q^2+4q^3+5q^4+4q^5+2q^6+q^7}{1+q+2q^2+2q^3+2q^4+q^5}$.

With this combinatorial model of fence posets, it is straightforward to deduce the degrees and valuations of numerators and denominators of q-rational numbers. Recall that we denote by $\varepsilon(x) = |a_1| + a_2 + \cdots + a_{2m} - 1$ so that $\varepsilon(x)+1$ is the number of vertices in the fence poset $\mathcal{F}_x$.

Corollary 1.4.65. *Let $x > 0$ be a positive rational number. With the same notations as above, the degrees and valuations of the polynomials deforming x are*

$$\begin{aligned}
&\deg(A^\sharp) = \varepsilon(x) && , \quad \deg(B^\sharp) = \varepsilon(x) - a_1, \\
&\mathrm{val}(A^\sharp) = \begin{cases} a_2 \text{ if } a_1 = 0 \\ 0 \text{ otherwise} \end{cases} && , \quad \mathrm{val}(B^\sharp) = 0, \\
&\deg(A^\flat) = \varepsilon(x) + 1 && , \quad \deg(B^\flat) = \varepsilon(x) - a_1 + 1, \\
&\mathrm{val}(A^\flat) = \begin{cases} a_2 \text{ if } a_1 = 0 \\ 0 \text{ otherwise} \end{cases} && , \quad \mathrm{val}(B^\flat) = 0.
\end{aligned}$$

The case of negative rational numbers can be computed using the symmetry $[-x]_q^\square = -q^{-1}[x]_{q^{-1}}^\square$.

We end this paragraph with the unimodality conjecture, proven by Kantarcı Oğuz and Ravichandran [KOR23] using the fence poset model.

Definition 1.4.66. Let $P = \sum_{i=0}^{d} x_i q^i$ be a polynomial. We say that P is *unimodal* when there is an index ℓ such that

$$x_0 \leqslant x_1 \leqslant \cdots \leqslant x_\ell \geqslant x_{\ell+1} \geqslant \cdots \leqslant x_d.$$

Theorem 1.4.67 ([KOR23])**.** *Numerators and denominators of right q-deformed rational numbers are unimodal.*

Remark 1.4.68. This theorem does not hold for left q-deformed rational numbers. Indeed, even left integers are not unimodal.

Remark 1.4.69. In the context of cluster algebras from surfaces, generating functions of ideals in fence posets appear as the specialization of coefficients variables to q of the F-polynomials in a surface without puncture. Recently, Kang, Lee and Lim extended the unimodality result for surfaces with punctures, corresponding to the co-called loop fence posets [KLL26].

1.5 The case of irrational numbers

In [MGO22], Morier-Genoud and Ovsienko proved some convergence properties on q - deformed rational numbers, leading to the definition of q-real numbers. These are formal power series in q. The first subsection introduces q-irrational numbers and their basic properties, in particular the injectivity of the quantization map on the real line, which was established in [Jou25b].

Two main questions have been explored so far in the literature on q-irrational numbers. First, what can one say about the radius of convergence of these q-series ? Second, what can one say about q-deformations of algebraic numbers ? We focus here on the second question, presenting the case of quadratic numbers in §1.5.2, that was worked out by Leclere and Morier-Genoud in [LMG21], and then giving partial answer for degree 4 and degree 6 algebraic numbers in §1.5.3 and §1.5.4 (these two are entirely based on [Jou25b]).

1.5.1 Definitions

Consider the ring $\mathbb{Z}[[q]][q^{-1}]$ of formal Laurent series in q,

$$\mathbb{Z}[[q]][q^{-1}] = \left\{ \sum_{k \geq \nu} x_k q^k \mid x_k \in \mathbb{Z},\ \nu \in \mathbb{Z} \right\},$$

endowed with its usual metric relying on the valuation ν of formal Laurent series,

$$\mathrm{dist}(f, g) = \frac{1}{2^{\nu(f-g)}}.$$

We present here only the notions immediately useful for our purpose. More precise definitions and properties about the ring $\mathbb{Z}[[q]][q^{-1}]$ and its role for q-real numbers can be found in [Lec24, Section 4.1.2].

Saying that a sequence $\left(x_n = \sum_{k \geq \nu_n} x_{n,k} q^k\right)_n$ in $\mathbb{Z}[[q]][q^{-1}]$ converges toward the series $y = \sum_{k \geq \nu} y_k q^k$ means that the valuations ν_n stabilize to ν and the coefficients $x_{n,k}$ stabilize to y_k as n grows,

$$\forall k \in \mathbb{Z}, \exists N \in \mathbb{N}, \forall n \geq N, \forall j \leq k, x_{n,j} = y_j.$$

Proposition 1.5.1 ([MGO22])**.** *Let $x \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number. Then for any sequence of rational numbers $(x_n)_n$ converging to x, the Taylor expansions of $[x_n]^\sharp$ converge toward a Laurent series that does not depend on the choice of the sequence $(x_n)_n$.*

Definition 1.5.2. With the previous notations, the series resulting of the convergence of the Taylor expansions of $([x_n]^\sharp)_n$ is defined to be the quantization of x and is denoted by $[x]_q$.

For irrational numbers, the left and right quantizations are the same.

Proposition 1.5.3 ([Jou25b])**.** *Let $x \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number. Let $(x_n)_n$ be a sequence of rational numbers converging to x. Then, the sequence $([x_n]^\sharp - [x_n]^\flat)_n$ converges to the zero series.*

Proof. We can suppose that $(x_n)_n$ is the sequence of convergents of x, $x_n = [a_0, a_1, a_2, \cdots, a_n]$, with $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{Z}_{\geq 1}$ for all $i \geq 1$.

Let us write, for all $n \in \mathbb{N}$, $[x_n]^\sharp = \pm q^d \frac{A_n(q)}{B_n(q)}$, with A_n and B_n two coprime monic polynomials in q with constant coefficient 1. By Corollary 1.4.65, the integer d depends on x as

$$d = \begin{cases} 0 \text{ if } x \geq 1 \\ \lfloor 1/x \rfloor \text{ if } 0 < x < 1 \\ \lfloor x \rfloor \text{ if } x < 0 \end{cases}.$$

The left version of x_n is then

$$[x_n]^\flat = \pm q^d \frac{qA_n(q) + (1-q)A_{n-1}(q)}{qB_n(q) + (1-q)B_{n-1}(q)},$$

so

$$[x_n]^\sharp - [x_n]^\flat = \pm q^d \frac{(1-q)(A_n(q)B_{n-1}(q) - A_{n-1}(q)B_n(q))}{qB_n(q)^2 + (1-q)B_n(q)B_{n-1}(q)}$$
$$= \frac{(1-q)q^{d+a_1+\cdots+a_n-1}}{qB_n(q)(B_n(q) - B_{n-1}(q)) + B_{n-1}(q)}.$$

And the series $1/(qB_n(q)(B_n(q) - B_{n-1}(q)) + B_{n-1}(q))$ starts with the coefficient 1 thanks to B_{n-1}, so the series $[x_n]^\sharp - [x_n]^\flat$ starts with $q^{d+a_1+\cdots a_n-1}$, and thus converges to the zero series when n grows. □

Corollary 1.5.4. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number. Let $(x_n)_n$ be a sequence of rational numbers converging to x. Then the sequence of left q-deformed numbers $([x_n]^\flat)_n$ converges to $[x]_q$.*

By convergence properties, the q-deformation of real numbers is equivariant with the action of $\mathrm{PGL}_2(\mathbb{Z})$.

Proposition 1.5.5 ([LMG21, Jou25b])**.** *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $M \in \mathrm{PGL}_2(\mathbb{Z})$. Then*

$$M_q \cdot [x]_q = [M \cdot x]_q.$$

Corollary 1.5.6. *For every irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$,*

$$[x+1]_q = q[x]_q + 1 \text{ and } [-x]_q = \frac{-[x]_q + 1 - q^{-1}}{(q-1)[x]_q + 1}.$$

The terminology of left and right deformations comes from the following.

Proposition 1.5.7 ([MGO22, BBL23])**.** *Let $x \in \mathbb{Q}$ be a rational number.*

(i) Let $(x_n)_n$ be a sequence of rational numbers converging to x by higher values (i.e. from the right of x). Then the sequences of right and left q-deformed numbers $([x_n]^\sharp)_n$ and $([x_n]^\flat)_n$ converge to $[x]^\sharp$.

(ii) Let $(x_n)_n$ be a sequence of rational numbers converging to x by lower values (i.e. from the left of x). Then the sequences of right and left q-deformed numbers $([x_n]^\sharp)_n$ and $([x_n]^\flat)_n$ converge to $[x]^\flat$.

We get two quantization maps, depending on which version one chooses for rational numbers.

$$Q^\square : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{Z}[[q]][q^{-1}] \\ x & \longmapsto & \begin{cases} [x]^\square \text{ if } x \in \mathbb{Q} \\ [x]_q \text{ else} \end{cases} \end{array}, \text{ with } \square \in \{\sharp, \flat\}.$$

Proposition 1.5.8 ([Jou25b])**.** *The two quantization maps $Q^\sharp$ and $Q^\flat$ are injective.*

Proof. We only prove the injectivity of $Q^\sharp$, the proof for the left version is the same.
On the rational numbers, $Q^\sharp$ is injective, because one can evaluate at $q = 1$ to recover the rational number from its deformation.
Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $x' \in \mathbb{R}$ be such that $[x]_q = [x']_q$. Note that if $x' \in \mathbb{Q}$, then $[x]_q$

is holomorphic in the neighbourhood of $q = 1$ and then $x = x'$. We suppose then that $x' \in \mathbb{R} \setminus \mathbb{Q}$. Consider the continued fractions of x and x':

$$x = [a_1, a_2, a_3, \cdots] \ , \ x' = [a'_1, a'_2, a'_3, \cdots].$$

By Corollary 1.4.65 (see also Theorem 2 of [MGO22]), a real number z is in $(0, 1)$ if and only if $[z]_q$ has a positive valuation. Since a_1 is the integer part of x, we can apply this to $z = T^{-a_1} \cdot x$. By equivariance property,

$$[T^{-a_1} \cdot x]_q = T_q^{-a_1} \cdot [x]_q = T_q^{-a_1} \cdot [x']_q = [T^{-a_1} \cdot x']_q.$$

Therefore $T^{-a_1} \cdot x'$ is in $(0, 1)$, and then a_1 is the integer part of x', i.e. $a_1 = a'_1$. Now put

$$y = \frac{1}{x - a_1} = [a_2, a_3, \cdots] \text{ and } y' = \frac{1}{x' - a_1} = [a'_2, a'_3, \cdots].$$

We have $y = JT^{-a_1} \cdot x$ and $y' = JT^{-a_1} \cdot x'$, so

$$[y]_q = [JT^{-a_1}]_q \cdot [x]_q = [JT^{-a_1}]_q \cdot [x']_q = [y']_q.$$

Again $a_2 = a'_2$, and by induction for all i, $a_i = a'_i$, so $x = x'$. □

1.5.2 Quadratic irrationals

The q-deformation of a quadratic irrational number can be written in a closed form, satisfying a q-deformed quadratic equation. This was proven in [LMG21] by Leclere and Morier-Genoud. This subsection summarizes their main results, to be compared with the generalizations to higher degrees in the next subsection.

Theorem 1.5.9 ([LMG21]). *Let z be an irrational number, root of $ax^2 + bx + c$, with $a, b, c \in \mathbb{Z}$. Then there exists $A, B, C \in \mathbb{Z}[q]$ such that $[z]_q$ satisfies*

$$A(q)[z]_q^2 + B(q)[z]_q + C(q) = 0.$$

Corollary 1.5.10. *Let $z = \frac{r \pm \sqrt{p}}{s}$ be an irrational quadratic number, with $r, s, p \in \mathbb{Z}$. Then there exist polynomials $R, S, P \in \mathbb{Z}[q]$ such that*

$$[z]_q = \frac{R(q) \pm \sqrt{P(q)}}{S(q)}.$$

Remark 1.5.11. Leclere and Morier-Genoud proved furthermore that the polynomial P in the above corollary is a palindrome. We will come back to this in Chapter 2.

The proof of Theorem 1.5.9 relies on the classical following fact.

Fact 1.5.12. *A real number z is a quadratic irrational number if and only if there exists a matrix $M \in \mathrm{SL}_2(\mathbb{Z})$, with $\mathrm{Tr}(M) > 2$, such that $M \cdot z = z$.*

Given such a matrix M fixing z, the equivariance property stated in Proposition 1.5.5 ensures that $M_q \cdot [z]_q = [z]_q$, and Theorem 1.5.9 then follows.

Example 1.5.13. Let us consider the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. It is solution of the quadratic equation $x^2 - x - 1 = 0$, and is a fixed point of the martix $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.
The q-version of M is $M_q = \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix}$, so $[\phi]_q$ is solution of the equation

$$qX^2 - (q^2 + q - 1)X - 1 = 0,$$

and $[\phi]_q = \frac{q^2+q-1+\sqrt{1+2q-q^2+2q^3+q^4}}{2q}$.
The golden ratio is part of a family of quadratic numbers called the metallic numbers, charaterized by the fact that their continued fraction expansion is periodic of period 1. The q-deformed metallic numbers were studied in [Ren22, OP25, HP25].

For higher degree algebraic numbers, the situation is more intricate as they are not anymore fixed points of fractional linear transformations. Still, there is a case where one can take advantage of the equivariance property again : this is when the algebraic number α and all its conjugates are related by fractional linear transformations. This idea was applied for cubic algebraic numbers in [OU25]. We give two other families of examples in the sequel.

1.5.3 Degree 4 algebraic numbers

This subsection is based on [Jou25b].

Consider the following particular case of algebraic equation of degree 4, with b an integer :

$$x^4 - bx^2 + 1 = 0. \tag{1.25}$$

We want to study the q-deformations of the solutions and the relations they share. Assuming there are four distinct real solution x_1, x_2, x_3 and x_4, we denote by σ_1, σ_2, σ_3 and σ_4 the elementary symmetric polynomials :

- $\sigma_1 = x_1 + x_2 + x_3 + x_4 = 0$;
- $\sigma_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = -b$;
- $\sigma_3 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = 0$;
- $\sigma_4 = x_1x_2x_3x_4 = 1$.

Notation 1.5.14. For $i \in \{1, 2, 3, 4\}$, the q-deformed number $[x_i]_q$ will be denoted X_i, and the elementary polynomials in the X_i's will be denoted by Σ_1, Σ_2, Σ_3, Σ_4. In general, these are Laurent series in q.

Lemma 1.5.15. *The equation (1.25) has four distinct real solutions if and only if $b > 2$. Moreover, the polynomial $x^4 - bx^2 + 1$ is irreducible over $\mathbb{Q}$ except when $b = n^2 \pm 2$, for $n \in \mathbb{N}$.*

Proposition 1.5.16 ([Jou25b])**.** *Let us suppose that $b > 2$ in equation (1.25). Then the q-deformed numbers X_1, X_2, X_3 and X_4 are solutions of the following quantized equation*

$$X^4 - \Sigma_1 X^3 + \left(\frac{q + q^{-1} - 3}{q - 1} \Sigma_1 + 2q^{-1} \right) X^2 + q^{-1}\Sigma_1 X + q^{-2} = 0,$$

where $\Sigma_1 = X_1 + X_2 + X_3 + X_4$ is a Laurent series in q.
In other words, the Vieta's relations become

- $\Sigma_4 = q^{-2}$,
- $\Sigma_3 = -q^{-1}\Sigma_1$,
- $(q-1)\Sigma_2 = (q+q^{-1}-3)\Sigma_1 + 2(1-q^{-1})$.

Proof. As the equation (1.25) is invariant under the two transformations $x \mapsto -x$ and $x \mapsto 1/x$, the matrices N and J act on its roots.

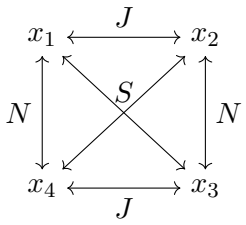


Figure 1.2: Galois group action on the roots by linear fractional transformations

After quantization, J_q and N_q also act on the X_i's, and then

$$\begin{pmatrix} -1 & 1-q^{-1} \\ q-1 & 1 \end{pmatrix} X_4 = X_1 \text{ so } (q-1)X_1X_4 = -X_1 - X_4 + 1 - q^{-1},$$

and same for X_2 and X_3 : $(q-1)X_2X_3 = -X_2 - X_3 + 1 - q^{-1}$. Letting J_q and S_q act, we get four more equations :

$qX_1X_2 = (q-1)(X_1+X_2)+1$, $qX_3X_4 = (q-1)(X_3+X_4)+1$, $qX_1X_3 = -1$ and $qX_2X_4 = -1$.

Using these equations, we get the quantized Vieta's relations. For instance,

$$\begin{aligned}(q-1)\Sigma_2 &= (q-1)(X_1X_2 + X_3X_4 + X_1X_4 + X_2X_3 + X_2X_4 + X_1X_3) \\ &= q^{-1}(q-1)^2(X_1+X_2+X_3+X_4) + 2q^{-1}(q-1) \\ &+ (-X_1 - X_2 - X_3 - X_4 + 2(1-q^{-1})) - 2q^{-1}(q-1) \\ &= (q-2+q^{-1})\Sigma_1 - \Sigma_1 + 2(1-q^{-1}) \\ &= (q-3+q^{-1})\Sigma_1 + 2(1-q^{-1}),\end{aligned}$$

and similarly for the other Vieta's relations. □

Remark 1.5.17. The Galois group of the polynomial in equation (1.25) is isomorphic to the group $\{\mathrm{Id}, J, S, N\}$ which is the Klein four group, acting on the roots x_1, x_2, x_3, x_4 exactly as shown in Figure 1.2.

Example 1.5.18. With $b = 4$, the roots are $\pm\sqrt{2 \pm \sqrt{3}}$. The q-deformation of $\sqrt{2+\sqrt{3}}$ is an infinite series in q, whose first terms are as follows:

$$\left[\sqrt{2+\sqrt{3}}\right]_q = 1+q^2-q^{15}+2q^{16}-2q^{17}+3q^{19}-3q^{20}+3q^{22}-3q^{23}+3q^{25}-3q^{26}+3q^{28}-4q^{29}...$$

The coefficient Σ_1 in the deformed version of the equation $x^4-4x+1=0$ is also an infinite Laurent series:

$$\Sigma_1 = -q^{-2} - 2q^{-1} + 1 + 2q - q^4 + 2q^5 - q^6 - q^8 + 2q^9 - q^{10} + q^{13} - 2q^{14} + 3q^{15}...$$

Example 1.5.19. Let us consider more specifically the case $b = n^2 - 2$, for $n \geq 3$. Then the polynomial splits in two quadratic polynomials

$$x^4 - (n^2-2)x^2 + 1 = (x^2 - nx + 1)(x^2 + nx + 1).$$

The solutions X_1, X_2, X_3, X_4 are then quadratic irrational numbers. Following [LMG21], these q-numbers are solutions in radicals of q-quadratic equations. For instance, let us detail the way $x^2 - nx + 1$ is being quantized. The two roots x_1, x_2 of this polynomial satisfy $x_i^2 = nx_i - 1$, so they are fixed points of the matrix $M = \begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix} = T^n S \in \mathrm{PSL}_2(\mathbb{Z})$. Then the q-deformed numbers X_1, X_2 are solutions of a quadratic equation given by the equality $M_q \cdot X_i = X_i$, where

$$M_q = T_q^n S_q = \begin{pmatrix} [n]^\sharp & -q^{n-1} \\ 1 & 0 \end{pmatrix}.$$

That is, the equation $x^2 - nx + 1 = 0$ is quantized by

$$X^2 - [n]^\sharp X + q^{n-1} = 0.$$

Likewise, the equation $x^2 + nx + 1$ is quantized by

$$X^2 - [-n]^\sharp X + q^{-n-1} = 0.$$

Therefore, the Vieta's formulas become :

$$\Sigma_4 = q^{-2} \ , \ \Sigma_3 = q^{n-1}[-n]^\sharp + q^{-n-1}[n]^\sharp,$$

$$\Sigma_2 = [n]^\sharp[-n]^\sharp + q^{-n-1} + q^{n-1} \text{ and } \Sigma_1 = [n]^\sharp + [-n]^\sharp.$$

The case $b = n^2 + 2$ is similar, except that here we will get a matrix with determinant -1. Indeed, the decomposition is

$$x^4 - (n^2+2)x^2 + 1 = (x^2 - nx - 1)(x^2 + nx - 1),$$

and the roots x_1, x_2 of $x^2 - nx - 1$ are fixed points of the matrix $\begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix} = NT^{-n}S$. Its deformation is

$$\begin{pmatrix} [n]^\flat & q^{-1} \\ 1 & q^{-1} - 1 \end{pmatrix}.$$

Therefore the quantized equations are

$$X^2 + (q^{-1} - 1 - [n]^\flat)X - q^{-1} = 0 \text{ and } X^2 + (q^{-1} - 1 - [-n]^\flat)X - q^{-1} = 0,$$

and the Vieta's relations for the equation $x^4 - (n^2+2)x^2 + 1 = 0$ become

$$\Sigma_4 = q^{-2} \ , \ \Sigma_3 = 2(q^{-2} - q^{-1}) - q^{-1}([n]^\flat + [-n]^\flat),$$

$$\Sigma_2 = [n]^\flat[-n]^\flat - (q^{-1}-1)([n]^\flat + [-n]^\flat) + 1 - 4q^{-1} + q^{-2} \text{ and } \Sigma_1 = 2(1-q^{-1}) + [n]^\flat + [-n]^\flat.$$

1.5.4 Degree 6 algebraic numbers

This subsection is based on [Jou25b].

Consider the following equation, depending in one integer parameter b :

$$x^6 - 3x^5 - bx^4 + (2b+5)x^3 - bx^2 - 3x + 1 = 0. \tag{1.26}$$

We keep the same notations as in the case of degree 4, the quantized solutions are denoted X_i $(1 \leq i \leq 6)$ and the symmetric polynomials Σ_i's.

Lemma 1.5.20. *The equation (1.26) has six distinct real solutions if and only if $b \geq 1$. In this case, it is irreducible over $\mathbb{Q}$ if and only if $b \neq 2$ and $b \notin \{k^2+k+1 \mid k \geq 0\}$.*

Proof. Set $f(x) = x^6 - 3x^5 - bx^4 + (2b+5)x^3 - bx^2 - 3x + 1$.
This polynomial is invariant under the action of J and of $\Gamma := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. So the set of roots is also stable by these transformations, and the roots are

$$x_1, \Gamma(x_1), \Gamma^2(x_1), 1/x_1, 1/\Gamma(x_1), 1/\Gamma^2(x_1).$$

Therefore either all the roots are real, either all the roots are nonreal complex numbers. Then it is straightforward to check that the roots are real if and only if $b \geq 1$.

Now let us suppose that $b \geq 1$. If the polynomial f is not irreducible over $\mathbb{Q}$, it has either two factors of degree 3 or three factors of degree 2.
• Let us suppose that f splits into two factors of degree 3. Then these two factors must be

$$(x - x_1)(x - \Gamma(x_1))(x - \Gamma^2(x_1)) \text{ and } \left(x - \frac{1}{x_1}\right)\left(x - \frac{1}{\Gamma(x_1)}\right)\left(x - \frac{1}{\Gamma^2(x_1)}\right).$$

So there is an integer a such that $x_1\Gamma(x_1) + x_1\Gamma^2(x_1) + \Gamma(x_1)\Gamma^2(x_1) = a$, that is

$$x_1^3 - (3+a)x_1^2 + ax_1 + 1 = 0,$$

so the polynomial f splits in two factors that are exaclty the one considered in [OU25],

$$f(x) = (x^3 - (3+a)x^2 + ax + 1)(x^3 + ax^2 - (3+a)x + 1) \quad (\star).$$

Identifying coefficient by coefficient, we get that a must satisfy $a^2+3a+3-b = 0$, therefore $4b-3$ must be a square integer, that is $b = k^2+k+1$, for some $k \in \mathbb{N}$.
Conversely, if $b = k^2+k+1$ for a $k \in \mathbb{N}$, then the factorization $(\star)$ works, with $a = k-1$.

• Let us suppose that f splits into three factors of degree 2. Then one can check that the only possible configuration is when $b = 2$ and

$$f(x) = (x^2 - 3x + 1)(x^2 - x - 1)(x^2 + x - 1).$$

Thus f is reducible over $\mathbb{Q}$ if and only if $b = 2$ or $b = k^2+k+1$ for some $k \in \mathbb{N}$. □

Proposition 1.5.21 ([Jou25b])**.** *Let us suppose that $b \geq 1$. After the q-deformation of the six roots of (1.26), the Vieta's relations are quantized as*

- $\Sigma_6 = 1$,
- $\Sigma_5 = -\Sigma_1 + 6$,
- $\Sigma_4 = -5\Sigma_1 + \Sigma_2 + 15$,
- $\Sigma_3 = -5\Sigma_1 + 2\Sigma_2 + 10$,
- $(q-1)\Sigma_2 = (q^2 + q - 4 + q^{-1})\Sigma_1 + 3(-q^2 + q + 2 - q^{-1})$.

Proof. We use the symmetries of (1.26), which is invariant under the action of J and Γ.

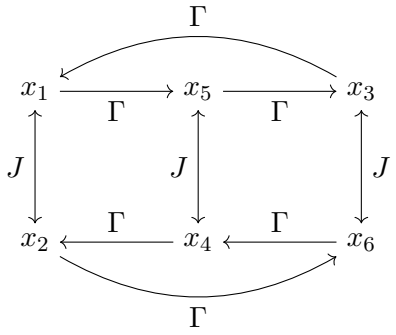


Figure 1.3: Galois group action on the roots by linear fractional transformations

After quantization, these transformations give

$$\begin{aligned}
X_1X_2 &= q^{-1}(q-1)(X_1+X_2)+q^{-1}, & X_1X_5 &= X_1 - 1,\\
X_4X_5 &= q^{-1}(q-1)(X_4+X_5)+q^{-1}, & X_3X_5 &= X_5 - 1,\\
X_3X_6 &= q^{-1}(q-1)(X_3+X_6)+q^{-1}, & X_1X_3 &= X_3 - 1,\\
X_5X_6 &= q(X_5+X_6)+1-q, & X_2X_4 &= X_4 - 1,\\
X_2X_3 &= q(X_2+X_3)+1-q, & X_4X_6 &= X_6 - 1,\\
X_1X_4 &= q(X_1+X_4)+1-q, & X_2X_6 &= X_2 - 1.
\end{aligned}$$

The q-deformed Vieta's relations follow. □

Example 1.5.22. For $b = 2$, the equation splits into $(x^2-3x+1)(x^2-x-1)(x^2+x-1) = 0$. The quantization of this equation comes from the quantization of quadratic equations, and it is

$$\left(X^2 - [3]^\sharp X + q^2\right)\left(qX^2 - (q^2+q-1)X - 1\right)\left(q^2X^2 - (q^2-q-1)X - q\right) = 0.$$

Therefore in this case, $\Sigma_1 = q + 2q^2 + 3q^3 + 2q^4 + q^5$ is finite, and we can check all the Vieta's relations of Proposition 1.5.21.

Example 1.5.23. Now let us take $b = 3$ so that the equation splits into

$$(x^3 - 3x^2 + 1)(x^3 - 3x + 1) = 0.$$

According to [OU25], these two factors are q-deformed as

$$(X^3 - B_1(q)X^2 + (B_1(q) - 3)X + 1)(X^3 - B_2(q)X^2 + (B_2(q) - 3)X + 1) = 0,$$

with $B_1(q) = X_1(q) + X_3(q) + X_5(q)$ (resp. $B_2(q) = X_2(q) + X_4(q) + X_6(q)$) the sum of odd (resp. even) roots.
We can check again that the relations of Proposition 1.5.21 are satisfied. Note that unlike the quadratic case, here it seems that the series $B_1(q)$ and $B_2(q)$ are actually infinite.

Remark 1.5.24. As all roots of the polynomial defining (1.26) are algebraic expressions of one (arbitrarily chosen) root x_1, the Galois group is isomorphic to the subgroup of $\mathfrak{S}_6$ generated by (1 2)(3 6)(4 5) and (1 3 5)(2 6 4), and its action on the roots is realized by the linear fractional transformations generated by J and Γ, as depicted in Figure 1.3.

Chapter 2

Traces and q-Markov numbers

> Eh quoi ! n'en pourrons-nous fixer au moins la trace ?
>
> *Le Lac, Méditations poétiques*,
> Alphonse de Lamartine

Markov numbers form a family of positive integers defined as solutions of the so-called Markov equation. They play a crucial role in number theory thanks to a famous theorem of Andrey Markov, enlightning how these numbers naturally appear in the theory of Diophantine approximation [Mar79]. Beside this major result, Markov numbers are also of interest in other fields such as combinatorics, hyperbolic geometry and mathematical physics. They were also extensively studied in order to answer a uniqueness conjecture, first formulated by Frobenius in 1913 [Fro13], and still open.

When the q-rational numbers of Morier-Genoud and Ovsienko were defined in 2020, Kogiso applied this construction to Markov numbers and defined the notion of q-Markov numbers [Kog20] in which we are interested in this chapter. Shortly after, Labbé and Lapointe studied the combinatorics of a different (but close) version of q-deformed Markov numbers [LL22]. A follow-up paper by Labbé, Lapointe and Steiner came few years later [LLS24], and very recently Aval and Labbé gave an interpretation of this version in terms of perfect matchings on snake graphs [AL25]. On the side of the version of q-Markov numbers introduced by Kogiso, a combinatorial interpretation has also been found, using circular fence posets, by Kantarcı Oğuz [KO25]. With Sam Evans, Sophie Morier-Genoud and Valentin Ovsienko, we contributed recently to this story by proving uniqueness results and another combinatorial interpretation of q-Markov numbers, based on perfect matchings in snake graphs [EJMGO26]. In the end of Chapter 3, we will describe yet another q-deformation of Markov numbers, still using the theory of q-rational numbers.

The idea of studying q-analogues and generalizations of Markov numbers is not new. Without pretending to be exhaustive, we outline some generalizations of Markov numbers, different from our approach with q-numbers, that can be found in the vast literature on the subject. A natural direction of generalization is to consider similar Diophantine equations and study their solutions, see [Cus93] and references therein. Another direction is to change the base ring of solutions to the Markov equation [Bow98, Mar25, CV25].

Moreover, with tools coming from Euclidean geometry (integer lattice geometry actually), Karpenkov and Van-Son defined a new notion of generalized Markov theory, extending the classical Markov's theorem. In 2026, Karpenkov and Ma established a combinatorial model based on weighted perfect matchings in generalized snake graphs to describe the geometric Markov numbers, and they extended this notion in a more general setting [KM26]. We can also find hyperbolic geometry methods in relation to Markov numbers, see the work of Springborn [Spr18] and its historical introduction. Using algebraic geometry and motivated by a link between Markov numbers and exceptional bundles on the projective plane [Rud89], Cotti and Varchenko studied a deformation of the Markov equation in [CV22]. This link was also exploited by Veselov [Ves25].

The connection between Markov numbers and the recent cluster algebra theory lead to other kinds of generalizations. We can mention the work of Lee, Li, Rabideau, Schiffler [LLRS23], or even more recently the generalized Markov equations introduced by Gyoda and Matsushita [GM23] and studied by Banaian, Gyoda [BG25], see also [BS24, GMS25]. Furthermore, a noncommutative version of the Markov equation was considered by Greenberg, Kaufman, Wienhard [GKW25]. Finally, a super-analogue of the Markov equation was defined by Huang, Penner, Zeitlin [HPZ23] in the context of higher Teichmüller theory, and it was investigated with a cluster algebraic language by Musiker [Mus25]. We generalized this super-equation in a joint work with Léa Bittmann, Ezgi Kantarcı Oğuz, Melody Molander and Emine Yıldırım [BJO$^+$26]. We also mention a different super-analogue of the Markov equation based on the notion of shadow sequences, introduced by Bonin and Ovsienko [BO23].

Organization and main results of Chapter 2.
We are interested in a deformation of Markov numbers constructed via the trace of some q-matrices introduced in Chapter 1. We begin in Section 2.1 by a general study of traces of q-matrices, that we will use in Section 2.3 to define a q-deformation of Markov numbers. The Section 2.2 is devoted to the classical theory of Markov numbers. We conclude the chapter by the short Section 2.4 about a relationship with the cluster algebra theory.

Section 2.1 comes from [Jou25b] and also contains several results from the former literature on traces of q-matrices [LMG21, KO25]. Section 2.2 consists of well-known facts about Markov numbers. The whole Section 2.3 is adapted from a joint work with Sam Evans, Sophie Morier-Genoud and Valentin Ovsienko [EJMGO26]. The final section is based on [EJMGO26] as well as on a private communication with Esther Banaian.

There are two main contributions gathered in this chapter. First, we study traces of q-matrices of determinant -1, and we get

- Theorem 2.1.2, traces of q-matrices are positive, palindromic polynomials.

Second, we define a notion of q-Markov numbers and prove in particular

- Proposition 2.3.3, ensuring that the q-Markov numbers defined by traces of q-Cohn matrices do not depend on the choice of the Cohn matrices,

- Theorem 2.3.10 which states that q-Markov numbers are the only solutions to a q-deformed Markov equation,

- Theorem 2.3.26, giving an analogue of the Frobenius uniqueness conjecture for our q-deformed Markov numbers.

Contents

2.1 Traces of q-matrices

2.1.1 Short reminders on q-matrices

We briefly recall important definitions about q-matrices. For a more complete exposition, see Sections 1.2 and 1.3 of Chapter 1.

Consider the q-deformations of the three usual generators of the group $\mathrm{PGL}_2(\mathbb{Z})$,

$$T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} , S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} , N_q = \begin{pmatrix} -1 & 1-q^{-1} \\ q-1 & 1 \end{pmatrix}. \tag{2.1}$$

They are elements of the projective group $\mathrm{PGL}_2(\mathbb{Z}[q, q^{-1}, t_q^{-1}])$, with

$$t_q := 1 - q + q^2,$$

so as such, they are defined up to a power of q, a power of t_q and up to sign.

Since there is a group isomorphism $\mathrm{PGL}_2(\mathbb{Z}) \simeq \langle T_q, S_q, N_q \rangle$, any matrix $M \in \mathrm{PGL}_2(\mathbb{Z})$ has a q-version $M_q \in \langle T_q, S_q, N_q \rangle =: \mathrm{PGL}_{2,q}(\mathbb{Z})$, obtained by replacing each T by T_q, each S by S_q and each N by N_q in any decomposition of M on the generators.

For example, the matrices $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ have q-versions

$$L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix} \text{ and } J_q = \begin{pmatrix} q-1 & 1 \\ q & 1-q \end{pmatrix}.$$

We distinguish three special q-matrices attached to a rational number $x \in \mathbb{Q}$, computed according to the three classical continued fraction expansions of x. If $x = [a_1, \cdots, a_{2m}]$ is the even positive continued fraction expansion of x, we consider

$$M_{+,q}(x) := T_q^{a_1} L_q^{a_2} \cdots T_q^{a_{2m-1}} L_q^{a_{2m}}.$$

If $x = [a_1, \cdots, a_{2m+1}]$ is the odd positive continued fraction expansion of x, we consider

$$M_{+,q}^{\mathrm{odd}}(x) := T_q^{a_1} J_q T_q^{a_2} J_q \cdots T_q^{a_{2m+1}} J_q.$$

If $x = [\![c_1, c_2, \cdots, c_k]\!]$ is the negative continued fraction expansion of x, we consider

$$M_{-,q}(x) := T_q^{c_1} S_q T_q^{c_2} S_q \cdots T_q^{c_k} S_q.$$

2.1.2 Positivity and palindromicity of traces

Definition 2.1.1. Let $f(q) \in \mathbb{Z}[q]$ be a polynomial. We say that $f(q)$ is palindromic when its sequence of coefficients forms a palindrome, i.e. if $q^{\deg(f)} f(q^{-1}) = f(q)$.

Theorem 2.1.2 ([LMG21, Jou25b])**.** *Let $M \in \mathrm{PGL}_2(\mathbb{Z})$. Then $\mathrm{Tr}(M_q)$ is a palindromic polynomial with positive integer coefficients, up to sign, q and t_q.*

We divide the proof of this theorem in two parts, the palindromicity on one hand and the positivity on the other hand. We adapt the proofs of Leclere and Morier-Genoud [LMG21] that considered matrices of determinant 1. In particular, we fix a minor error in their statement of positivity.

We will use the following notation, generalizing the matrix $M_-(x)$ defined in Chapter 1, Definition 1.7.

Notation 2.1.3. Let $c_1, c_2, \cdots, c_k$ be integers. We denote

$$M(c_1, c_2, \cdots, c_k) := T^{c_1} S T^{c_2} S \cdots T^{c_k} S \in \mathrm{PSL}_2(\mathbb{Z}).$$

We also denote by $M_q(c_1, c_2, \cdots, c_k)$ the q-deformed version of this matrix (we choose the representant in $\mathrm{GL}_2(\mathbb{Z}[q, q^{-1}, t_q^{-1}])$ given by the q-generators as in (2.1)).

Proof of the palindromicity. If $\det(M) = 1$, the palindromicity of $\mathrm{Tr}(M_q)$ is a result of Leclere and Morier-Genoud [LMG21, Theorem 3].
If $\det(M) = -1$, there are integers $c_1, c_2, \cdots, c_k \in \mathbb{Z}$ such that $M = NT^{c_1} S T^{c_2} S \cdots T^{c_k} S = NM(c_1, c_2, \cdots, c_k)$. Let us compute ${}^t M_{q^{-1}}$.

$$\begin{aligned}
{}^t M_{q^{-1}} &= \begin{pmatrix} [c_k]_{q^{-1}}^\sharp & 1 \\ -q^{1-c_k} & 0 \end{pmatrix} \cdots \begin{pmatrix} [c_2]_{q^{-1}}^\sharp & 1 \\ -q^{1-c_2} & 0 \end{pmatrix} \begin{pmatrix} [c_1]_{q^{-1}}^\sharp & 1 \\ -q^{1-c_1} & 0 \end{pmatrix} {}^t N_{q^{-1}} \\
&= q^{-\Sigma(c_i-1)} \begin{pmatrix} [c_k]_q^\sharp & q^{c_k-1} \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} [c_2]_q^\sharp & q^{c_2-1} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} [c_1]_q^\sharp & q^{c_1-1} \\ -1 & 0 \end{pmatrix} {}^t N_{q^{-1}} \\
&= q^{-\Sigma(c_i-1)} N M_q(c_k, \cdots, c_2, c_1) N^t N_{q^{-1}},
\end{aligned}$$

because the conjugation by N turns the signs of the antidiagonale to the opposite. And $N^t N_{q^{-1}} = N_q N$, so taking the trace we get :

$$\mathrm{Tr}(M_{q^{-1}}) = q^{-\Sigma(c_i-1)} \mathrm{Tr}\left(M_q(c_k, \cdots, c_2, c_1) N_q\right).$$

It is thus enough to show that $\mathrm{Tr}(M_q(c_k,\cdots,c_2,c_1)N_q) = \mathrm{Tr}(N_qM_q(c_1,c_2,\cdots,c_k))$. Let us set

$$M_q(c_1,c_2,\cdots,c_k) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } M_q(c_k,\cdots,c_2,c_1) = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}.$$

With these notations, $\mathrm{Tr}(N_qM_q(c_1,c_2,\cdots,c_k)) = -A + (1-q^{-1})C + (q-1)B + D$, and same for $\mathrm{Tr}(N_qM_q(c_k,\cdots,c_2,c_1))$ with $\tilde{A},\tilde{B},\tilde{C},\tilde{D}$. In the proof of Lemma 3.8 of [LMG21], the following relations are settled, using induction on the length of the continued fraction $[\![c_1,\cdots,c_k]\!]$ and on the integer c_k :

$$\begin{cases} A + D = \tilde{A} + \tilde{D} & (1) \\ C - qB = \tilde{C} - q\tilde{B} & (2) \\ A + B - C = \tilde{A} + \tilde{B} - \tilde{C} & (3) \end{cases}.$$

Taking (2) + (3), we get
$A - (q-1)B = \tilde{A} - (q-1)\tilde{B}$, and taking (1) $- q^{-1}$(2) $-$ (3), we get $D + (1-q^{-1})C = \tilde{D} + (1-q^{-1})\tilde{C}$, and this shows exactly that

$$\mathrm{Tr}(N_qM_q(c_1,c_2,\cdots,c_k)) = \mathrm{Tr}(N_qM_q(c_k,\cdots,c_2,c_1)).$$

Therefore $\mathrm{Tr}(M_{q^{-1}}) = q^{-\Sigma(c_i-1)}\,\mathrm{Tr}(M_q)$, meaning that $\mathrm{Tr}(M_q)$ is a palindromic polynomial.

Proof of the positivity. Because the Theorem 2.1.2 states a result up to powers of q and sign, in the rest of the paragraph we work up to these. To emphasize this convention, we will write $A \equiv B$ when two Laurent polynomials A and B are equal up to a power of q and a sign.
Besides, we write $A \geq 0$ to say that the Laurent polynomial A has positive coefficients.

We begin with the particular case of a matrix $M_-(x)$. This lemma is a corrected version of Lemma 3.10 in [LMG21].

Lemma 2.1.4. *Let $x = [\![c_1,c_2,\cdots,c_k]\!]$ be a rational number, with $x \geq 1$ so that the $c_i's$ are integers greater than 2. Then $\mathrm{Tr}(M_{-,q}(x)) \geq 0$.*

Proof. We start an induction on k, aiming to prove that

$$(H) \quad \forall k \geq 1,\ \ \forall (c_1,\cdots,c_k) \in \mathbb{Z}^k_{\geq 2}, \text{ if } M_q(c_1,\cdots,c_k) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ we have}$$

$$A \geq 0\ ,\ -B \geq 0\ ,\ A + D \geq 0 \text{ and } A + B \geq 0.$$

For $k = 1$,

$$M_q(c_1) = \begin{pmatrix} [c_1]^\sharp & -q^{c_1-1} \\ 1 & 0 \end{pmatrix}.$$

And (H) is satisfied.

Let $k \geq 2$ and let us suppose that (H) is true for any $k_0 \leq k$. Let $c_1,\cdots,c_k,c$ be $k+1$ integers greater or equal to 2. We set

$$M_q(c_1,\cdots,c_k) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

so that by hypothesis we have $A \geq 0$, $-B \geq 0$, $A + D \geq 0$ and $A + B \geq 0$.
Moreover, as the c_i's are greater than 2, the rational number $x = [\![c_1, c_2, \cdots, c_k]\!]$ is greater than 1, and by total positivity of q-rational numbers, we can deduce that $A - C \geq 0$.

One can compute

$$M_q(c_1, \cdots, c_k, c) = \begin{pmatrix} [c]^\sharp A + B & -q^{c-1}A \\ [c]^\sharp C + D & -q^{c-1}C \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}.$$

Then $A' = q[c-1]^\sharp A + (A + B) \geq 0$, $-B' = q^{c-1}A \geq 0$, and

$$\begin{aligned} A' + D' &= q[c-1]^\sharp A + A + B - q^{c-1}C \\ &= q[c-2]^\sharp A + q^{c-1}(A - C) + (A + B) \\ &\geq 0. \end{aligned}$$

Likewise,

$$\begin{aligned} A' + B' &= [c]^\sharp A + B - q^{c-1}A \\ &= q[c-2]^\sharp A + (A + B) \\ &\geq 0. \end{aligned}$$

This concludes the induction and the proof of the lemma. □

In general, a matrix M with $\det(M) = 1$ can be written as $M = M(c_1, c_2, \cdots, c_k)$, and Proposition 7.4 of [MGO19] ensures that we can suppose $c_3, \cdots, c_{k-2} \geq 2$, and $1 \leq c_1 \leq c_2$, and $1 \leq c_k \leq c_{k-1}$. The trace of $M_q(c_1, c_2, \cdots, c_k)$ is invariant by circular permutation of the c_i's, and $M_q(1, 1, 1) = -q^3\,\mathrm{Id}$, so we can even suppose that $c_2, \cdots, c_k \geq 2$ and $1 \leq c_1$.

• Case 1 : $c_1 \geq 2$. By Lemma 2.1.4, $\mathrm{Tr}(M_q)$ has positive coefficients.

• Case 2 : $c_1 = 1$ and $k = 1$ or $c_1 = 1$, $c_2 = 2$ and $k = 2$. Then we can check that $\mathrm{Tr}(T_q S_q) = q$ and $\mathrm{Tr}(T_q S_q T_q^2 S_q) = 0$ have positive coefficients.

• Case 3 : $c_1 = 1$ and $k \geq 2$. Using the relation $T_q S_q T_q = -q S_q T_q^{-1} S_q$ and circular permutations, we have $\mathrm{Tr}(M_q) \equiv \mathrm{Tr}(M_q(c_2 - 1, c_3, \cdots, c_{k-1}, c_k - 1))$, so by induction on k the result follows.

It remains to consider the case of a matrix M with $\det(M) = -1$. Again, we begin with a particular case.

Lemma 2.1.5. *Let $x = [\![c_1, c_2, \cdots, c_k]\!] \in \mathbb{Q}_{\geq 0}$, so $c_i \geq 2$ for $i \in [\![2, k]\!]$ and $c_1 \geq 1$. Then $-\mathrm{Tr}(N_q M_{-,q}(x)) \geq 0$.*

Proof. Denote $[x]^\sharp = \frac{A}{B}$ with A and B two coprime polynomials. Since $x \geq 0$, the denominator B has constant coefficient 1. Let the left q-deformation of $-x$ be $\frac{-A^\flat}{B^\flat}$, with $A^\flat$ and $B^\flat$ two coprime polynomials. Since $-x \leq 0$, the numerator $-A^\flat$ has constant coefficient

-1. By Theorem 1.3.6 applied to N_q, we have $N_q \cdot \frac{A}{B} = \frac{-qA+(q-1)B}{(q^2-q)A+qB} = \frac{-A^\flat}{B^\flat}$, and matching valuations we get

$$-A^\flat = -qA + (q-1)B \text{ and } B^\flat = (q^2-q)A + qB.$$

Then by Proposition 1.4.6,

$$N_q M_q(c_1, c_2, \cdots, c_k) = q^{k-1} \begin{pmatrix} -A^\flat & q^{c_k-1}A^\flat_{k-1} \\ B^\flat & -q^{c_k-1}B^\flat_{k-1} \end{pmatrix},$$

where $\frac{-A^\flat_{k-1}}{B^\flat_{k-1}}$ is the left q-deformation of $-x' = -[\![c_1, c_2, \cdots, c_{k-1}]\!]$.
By the positivity property of left q-rationals, $A^\flat$ and $B^\flat_{k-1}$ have positive coefficients, and then $\mathrm{Tr}(M_q) = -q^{k-1}A^\flat - q^{k+c_k-2}B^\flat_{k-1}$ has negative coefficients. □

In general, a matrix M with determinant -1 can be written as $M = NM(c_1, c_2, \cdots, c_k)$, with $c_3, \cdots, c_{k-2} \geq 2$ and $1 \leq c_1 \leq c_2$, $1 \leq c_k \leq c_{k-1}$. To manage circular permutations, we will use $M_q(c)N_q = -q^c N_q M_q(-c)$, implying that

$$\mathrm{Tr}(N_q M_q(c_1, c_2, \cdots, c_k)) \equiv \mathrm{Tr}(N_q M_q(c_2, \cdots, c_k, -c_1)).$$

We start an induction on k. If $k = 1$ or $k = 2$, direct computations show the result. Suppose then $k \geq 3$.
• Case 1 : $c_k \geq 2$. If $c_2 \geq 2$, the Lemma 2.1.5 above applies. Suppose $c_2 = c_1 = 1$. We can apply relation $T_qS_qT_qS_qT_q = q^2S_q$, and get $\mathrm{Tr}(M_q) \equiv \mathrm{Tr}(N_qS_qM_q(c_3 - 1, \cdots, c_k))$. If $k = 3$, we can conclude easily, and if $k \geq 4$, acording to whether $c_k \geq c_3 - 1$ or not and using circular permutation, we get $\mathrm{Tr}(N_qM_q(c_4, \cdots, c_{k-1}, c_k - c_3 + 1))$ or $\mathrm{Tr}(N_qM_q(c_3 - c_k - 1, c_4, \cdots, c_{k-1}))$ and we conclude by induction on k.

• Case 2 : $c_{k-1} \geq 2$ and $c_k = 1$. By applying $S_qT_qS_q = q^2T_q^{-1}S_qT_q^{-1}$ and a circular permutation, we get $\mathrm{Tr}(M_q) = \mathrm{Tr}(N_qM_q(c_1 + 1, c_2, \cdots, c_{k-2}, c_{k-1} - 1))$ and we conclude by induction.

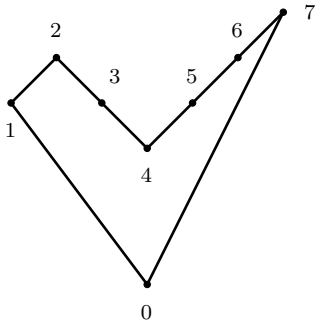
• Case 3 : $c_{k-1} = c_k = 1$.
By $(T_qS_q)^3 = -q^3\,\mathrm{Id}$, we have $M_q(c_1, \cdots, c_k) = M_q(c_1, \cdots, c_{k-3})T_q^{c_{k-2}-1}$. Thus, according to whether $c_1 \geq c_{k-2} - 1$ or not, we apply a circular permutation to get $\mathrm{Tr}(N_qM_q(c_1 - c_{k-2} + 1, c_2, \cdots, c_{k-3}))$ or $\mathrm{Tr}(N_qM_q(c_2, \cdots, c_{k-2} - 1 - c_1))$ and conclude by induction.

2.1.3 Combinatorial interpretation of traces

Matrices with determinant 1

Definition 2.1.6 (Closure of a fence poset)**.** Let $\mathcal{F}$ be a fence poset, with vertices $v_0, v_1, \cdots, v_{N-1}$. The closure of $\mathcal{F}$, denoted by $\overline{\mathcal{F}}$, is the fence poset obtained from $\mathcal{F}$ by adding the cover relation $v_0 \lhd v_{N-1}$. The closure is then called a circular poset.

Example 2.1.7. Recall the fence poset associated to $\frac{22}{9}$, displayed in Example 1.4.59. The Hasse diagram of its closure $\overline{\mathcal{F}_{\frac{22}{9}}}$ is displayed below. To save space, we label the vertex v_i by i.



The positivity of traces of some q-matrices can be interpreted in a combinatorial setting. Let $x \in \mathbb{Q}_{\geq 1}$. Recall the construction of the fence poset $\mathcal{F}_x$ associated to x, as in Definition 1.4.56.

Consider the two distinguished matrices $M_{+,q}(x)$ and $M_{-,q}(x) \in \mathrm{PSL}_{2,q}(\mathbb{Z})$ associated to x, as in Definition 1.20 (see also reminders in the beginning of the section).

Proposition 2.1.8 ([KO25])**.** *The trace* $\mathrm{Tr}(M_{+,q}(x))$ *is the generating function of ordered ideals in the closure of* $\mathcal{F}_x$*, and* $\mathrm{Tr}(M_{-,q}(x))$ *is the generating function of ordered ideals in the closure of* $\mathcal{F}_x \setminus \{v_0\}$*.*

$$\mathrm{Tr}(M_{+,q}(x)) = \sum_{I \text{ ordered ideal in } \overline{\mathcal{F}_x}} q^{|I|},$$

$$\mathrm{Tr}(M_{-,q}(x)) = \sum_{I \text{ ordered ideal in } \overline{\mathcal{F}_x \setminus \{v_0\}}} q^{|I|}.$$

Since all matrices in $\mathrm{PSL}_{2,q}(\mathbb{Z})$ have a trace with positive coefficients (up to sign), one could ask for an analogue interpretation in general. In the previous paragraph, we saw that the trace of any element M of $\mathrm{PSL}_{2,q}(\mathbb{Z})$ is either zero, or q, or equal to the trace of some $M_q(c_1, c_2, \cdots, c_n)$, with all the c_i's greater than 2. In that sense, the proposition above covers any interesting case. However, this does not provide an explicit construction directly from the data of the element M. We give here a partial answer to this question.

Definition 2.1.9. Let $\mathcal{F}$ be a fence poset, with vertices v_0, v_1, ... v_{N-1} and suppose that the first covering relation is $v_0 \lhd v_1$. Let $k \in \mathbb{Z}_{\geq 0}$. Define the enriched fence poset of degree k to be the fence poset obtained from $\mathcal{F}$ by adding k vertices w_1, ... w_k with covering relations $v_0 \lhd w_1 \lhd w_2 \lhd \cdots \lhd w_k \lhd v_1$. Denote this new fence poset by $\mathcal{F}^{(k)}$.

Definition 2.1.10. Let $\mathcal{F}$ be a fence poset, with vertices v_0, v_1, ... v_{N-1}. Let α be the maximum index such that $v_0 \leq v_\alpha$. Let $k \in [\![0, \alpha]\!]$. Define the impoverished fence poset of degree k to be the fence poset obtained from $\mathcal{F}$ by deleting the first k vertices v_0, ... v_{k-1}. Denote this new fence poset by $\mathcal{F}^{(-k)}$.

Proposition 2.1.11. *Let* $M \in \mathrm{PSL}_2(\mathbb{Z})$*, and let* $x \in \mathbb{Q}$ *such that* $M \cdot \frac{1}{0} = x$*. We suppose that* $x \geq 1$ *and denote* $\alpha := \lfloor x \rfloor$*. Then*

(i) There is an integer $k \in \mathbb{Z}$ *such that* $M = M_{+,q}(x)T^k$*.*

(ii) If $-\alpha \leq k$*, the trace* $\mathrm{Tr}(M_q)$ *is the generating function of ordered ideals in* $\overline{\mathcal{F}_x^{(k)}}$*.*

Proof. Both M and $M_+(x)$ send $\frac{1}{0}$ to x, so $M_+(x)^{-1}M$ is in the stabilizer of $\frac{1}{0}$, which is the group generated by T. This proves the first point (i).
Let $x = [a_1, a_2, \cdots, a_{2m}]$ be the even positive continued fraction expansion of x, and let $x' = [a_1, a_2, \cdots, a_{2m-1}]$. Denote $[x]^\sharp = \frac{A}{B}$ and $[x']^\sharp = \frac{A'}{B'}$. We can compute M_q using Proposition 1.4.6.

$$M_q = M_{+,q}(x)T_q^k = \begin{pmatrix} qA & A' \\ qB & B' \end{pmatrix} \begin{pmatrix} q^k & [k]_q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q^{k+1}A & q[k]A + A' \\ q^{k+1}B & q[k]B + B' \end{pmatrix}.$$

Hence the trace is $\mathrm{Tr}(M_q) = q^{k+1}A + q[k]B + B'$.
Suppose first that $k \geq 0$, and let us enumerate ordered ideals in $\overline{\mathcal{F}_x^{(k)}}$, by dividing them in two groups.

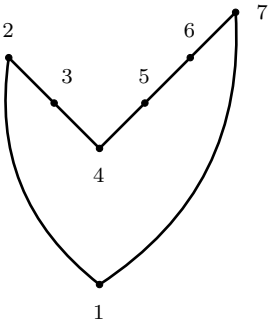
- Group 1 : ordered ideals I such that $v_0 \notin I$. They can not contain any of the vertices w_i, neither $v_1, \cdots, v_{a_1}$, nor $v_{N-1}, \cdots, v_{N-a_{2m}}$. Therefore they are in bijection with ordered ideals in $\mathcal{F}_{x'} \setminus \{v_0, v_1, \cdots, v_{a_1}\}$, which is counted by B'.
- Group 2 : ordered ideals I such that $v_0 \in I$. They are in bijection with ordered ideals containing the first vertex in the fence poset $\mathcal{F}_y$, with $y = [a_1 + k, a_2, \cdots, a_{2m}]$. By Corollary 1.4.63, this is enumerated by the numerator of $[y]_q$.
  But $y = x + k = T^k \cdot x$, and by equivariance $[y]_q = \frac{q^k A + [k]B}{B}$. Therefore this group is enumerated by $q(q^k A + [k]B)$ (the multiplication by q stands for the presence of v_0 in these ideals).

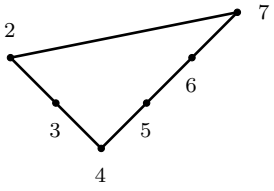
This proves that the formula $q^{k+1}A + q[k]B + B'$ is the generating function of $\overline{\mathcal{F}_x^{(k)}}$.
Now suppose $-\alpha \leq k < 0$. In the circular impoverished poset $\overline{\mathcal{F}_x^{(k)}}$, the first vertex is v_{-k}, with cover relations $v_{-k} \lhd v_{-k+1}$ and $v_{-k} \lhd v_{N-1}$. The enumeration can be adapted for this case by replacing the role of v_0 by v_{-k}, and the proof works then exactly the same. □

Example 2.1.12. Consider our favorite rational number $x = \frac{22}{9}$. The associated negative matrix is $M_-(x) = M_+(x)T^{-1} = \begin{pmatrix} 22 & -5 \\ 9 & -2 \end{pmatrix}$, and

$$\mathrm{Tr}\left(M_{-,q}\left(\frac{22}{9}\right)\right) = \mathrm{Tr}(M_{+,q}(x)T_q^{-1}) = 1 + 2q + 3q^2 + 4q^3 + 4q^4 + 3q^5 + 2q^6 + q^7.$$

This is the generating function of ordered ideals in the circular impoverished poset $\overline{\mathcal{F}_x^{(-1)}}$ displayed below, on the left.

$\overline{\mathcal{F}_{\frac{22}{9}}^{(-1)}} =$ 

$\overline{\mathcal{F}_{\frac{22}{9}}^{(-2)}} =$ 

Since the integer part of $\frac{22}{9}$ is 2, we can also consider $\mathrm{Tr}(M_{+,q}(x)T_q^{-2})$, which is associated to the circular impoverished poset $\overline{\mathcal{F}_x^{(-k)}}$ displayed on the right. The generating function is

$$\mathrm{Tr}(M_{+,q}(x)T_q^{-2}) = 1 + q + 2q^2 + 3q^3 + 2q^4 + q^5 + q^6.$$

Matrices with determinant -1

To give an interpretation of traces of q-matrices with determinant -1, we have to define a modified version of the closure of a fence poset, compatible with admissible ideals we defined for left q-rational numbers.

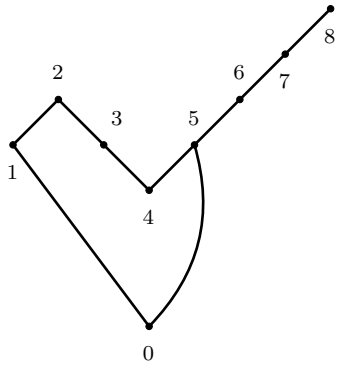
Definition 2.1.13. Let $x \in \mathbb{Q}_{\geq 1}$, and let $\mathcal{F}_x^\flat$ be the left fence poset associated to it. Let a_{2m+1} be the last part of the odd positive continued fraction of x.
The left closure of $\mathcal{F}_x^\flat$, denoted by $\overline{\mathcal{F}_x}^\flat$, is the fence poset obtained from $\mathcal{F}_x^\flat$ by adding the cover relation $v_0 \lhd v_{N-a_{2m+1}+1}$. We call this poset the left circular poset associated to x.

Definition 2.1.14. Let $x \in \mathbb{Q}_{\geq 1}$ and let $\overline{\mathcal{F}_x}^\flat$ be the left circular poset associated to x. Let a_{2m+1} be the last part of the odd positive continued fraction of x.
We say that an ideal I of $\overline{\mathcal{F}_x}^\flat$ is admissible when it satisfies the two conditions below.

I- If $v_0 \in I$, then $v_N \in I$ if and only if $v_{N-1} \in I$;

II- If $v_0 \notin I$, then $v_{N-a_{2m+1}} \in I$ if and only if $v_{N-a_{2m+1}-1} \in I$.

Example 2.1.15. With $x = \frac{22}{9} = [2, 2, 4]$, the left circular poset is



There is only one admissible ideal of size 1 in $\overline{\mathcal{F}_{\frac{22}{9}}}^\flat$, which is $\{0\}$.

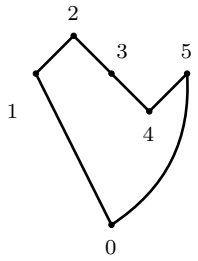
This notion of admissible ideal in circular poset is designed to match the combinatorics of left q-rational numbers. The following proposition is a direct consequence of Proposition 1.4.13 and Corollary 1.4.63

Proposition 2.1.16. *Let $x \in \mathbb{Q}_{\geq 1}$ and let $M_{+,q}^{\mathrm{odd}}(x) \in \mathrm{PGL}_{2,q}(\mathbb{Z})$ be the odd positive matrix associated to x, as in Definition 1.22. Up to a power of $t_q = 1 - q + q^2$, the trace of this matrix is the generating function of admissible ideals in the left circular poset $\overline{\mathcal{F}_x}^\flat$.*

$$\mathrm{Tr}(M_{+,q}^{\mathrm{odd}}(x)) = t_q^m \sum_{I \text{ admissible in } \overline{\mathcal{F}_x}^\flat} q^{|I|},$$

where $2m+1$ is the lenght of the odd positive continued fraction of x.

Example 2.1.17. Consider now the case of a rational number whose odd positive continued fraction ends with a 1 : $x = \frac{7}{3} = [2, 2, 1]$. The left circular fence poset $\overline{\mathcal{F}_{\frac{7}{3}}^{\flat}}$ is pictured below on the left, and admissible ideals on the right.



Size	Admissible ideals
≤ 2	$\{0\}$, $\{0,1\}$, $\{3,4\}$
3	$\{0,4,5\}$
4	$\{0,3,4,5\}$, $\{0,1,4,5\}$
5	$\{0,1,3,4,5\}$
6	$\{0,1,2,3,4,5\}$

This matches with of $M_{+,q}^{\text{odd}}\left(\frac{7}{3}\right) = (1-q+q^2)\begin{pmatrix} q+q^2+q^3+2q^4+q^5 & 1+q+q^2+q^3+q^4 \\ q+q^3+q^4 & 1+q^2 \end{pmatrix}$.

2.2 Classical Markov numbers and Cohn matrices

The material of this section is classical and can be found for example in [Aig13].

The Markov equation is the following Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz. \tag{2.2}$$

Definition 2.2.1. A *Markov triple* (a, b, c) is a triple of positive integers solution to the Markov equation (2.2).
A *Markov number* is a positive integer appearing in a Markov triple.

Example 2.2.2. The triple $(1, 5, 2)$ is a Markov triple.

Remark 2.2.3. Since the Markov equation is symmetric in x, y and z, any permutation of a given Markov triple (a, b, c) is again a Markov triple. For this reason, we will consider triples up to permutation of the coordinates, and often choose to put the maximum of the triple in second coordinate.

2.2.1 Markov tree and Stern-Brocot tree

From the trivial solution $(1, 1, 1)$, one can derive other solutions of the Markov equation by sequences of transformations called the Vieta jumps.

Definition 2.2.4. Let (a, b, c) be a Markov triple. A Vieta jump at a is the operation $(a, b, c) \mapsto (a', b, c)$ where

$$a' = \frac{b^2 + c^2}{a} = 3bc - a.$$

The Vieta jumps at b and c are defined similarly.

Proposition 2.2.5. *If (a, b, c) is a Markov triple, then (a', b, c), (a, b', c) and (a, b, c') obtained by Vieta jumps at a, b and c respectively are again Markov triples.*

Proof. Consider the quadratic polynomial $x^2 - 3bcx + b^2 + c^2$. Since (a, b, c) is a Markov triple, a is a root of this polynomial. By Vieta's formulas, the other root is $3bc - a = \frac{b^2+c^2}{a} = a'$, and then (a', b, c) is a solution of the Markov equation. □

Applying Vieta jump to $(1,1,1)$, one gets the Markov triple $(1,2,1)$. These are the only two Markov triples with a repeated coordinate. They are called the *singular* Markov triples.

Definition 2.2.6. The *Markov tree* is an infinite binary tree, rooted at $(1,5,2)$ and such that the two children of a node labeled (a,b,c) have labels (a,c',b) (on the left) and (b,a',c) (on the right).

Note that with this convention, the second coordinate of a triple in the Markov tree is always the maximum of the triple. Indeed, if $b = \max(a,b,c)$, then $c' = \frac{a^2+b^2}{c} > \frac{a^2}{b} + b > b$ and similarly for a'.

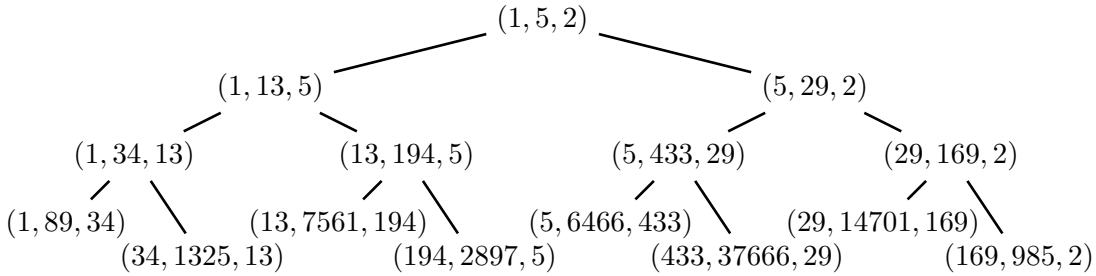


Figure 2.1: First levels of the Markov tree

The following statement is proven in [Aig13, Theorem 3.3].

Proposition 2.2.7. *The Markov tree contains every non singular Markov triple exactly once.*

The famous *uniqueness conjecture* stated by Frobenius in 1913 [Fro13] claims that any Markov number appears exactly once as a maximum of a Markov triple.

To work with the Markov tree, we use a parametrization by rational numbers between 0 and 1 given by the Stern-Brocot tree.

Definition 2.2.8. The *Stern-Brocot tree* is an infinite binary tree, rooted at $\left(\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right)$, and such that the two children of a node labeled (x,y,z) have labels $(x, x \oplus_F y, y)$ (on the left) and $(y, y \oplus_F z, z)$ (on the right), where $\oplus_F$ denotes the Farey sum (recall Definition 1.1.10).

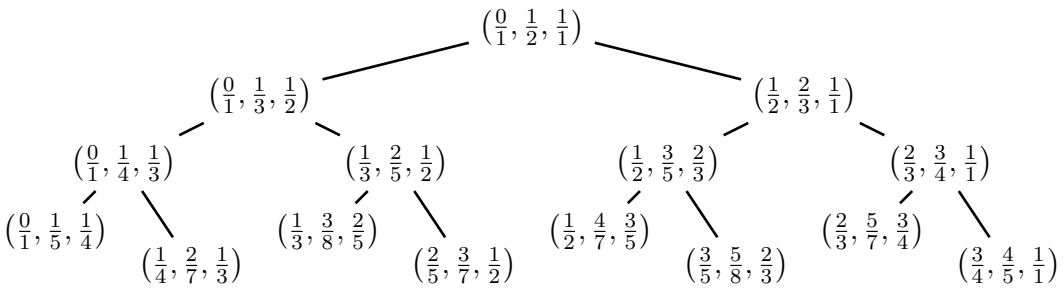


Figure 2.2: First levels of the Stern-Brocot tree

The Stern-Brocot tree can be understood as another presentation of the Farey graph (restricted to $[0,1]$). It contains every rational number between 0 and 1, exactly once as the middle coordinate of a triple (except for 0 and 1). The obvious bijection between the Stern-Brocot tree and the Markov tree provides an application assigning a Markov number to any rational between 0 and 1.

Definition 2.2.9. Let $t \in (0,1)$ be a rational number, and let (s,t,u) be the unique triple in Stern-Brocot tree whose middle coordinate is t. The associated Markov number m^t is the middle coordinate of the Markov triple appearing at the same place as (s,t,u) in the Markov tree.
For the sake of completeness, we also put $m^{\frac{0}{1}} = 1$ and $m^{\frac{1}{1}} = 2$.

Example 2.2.10. The left-most branch of the Markov tree is called the *Fibonacci branch* because it contains the odd Fibonacci numbers. In the Stern-Brocot tree, this is the branch of the inverses of positive integers $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots$. Hence

$$m^{\frac{1}{1}} = 2,\ m^{\frac{1}{2}} = 5,\ m^{\frac{1}{3}} = 13,\ m^{\frac{1}{4}} = 34,\ m^{\frac{1}{5}} = 89, \cdots$$

Example 2.2.11. Another remarkable branch is the right-most branch of the Markov tree. It contains odd Pell numbers and is then called the *Pell branch*.

$$m^{\frac{1}{2}} = 5,\ m^{\frac{2}{3}} = 29,\ m^{\frac{3}{4}} = 169,\ m^{\frac{4}{5}} = 985,\ m^{\frac{5}{6}} = 5741, \cdots$$

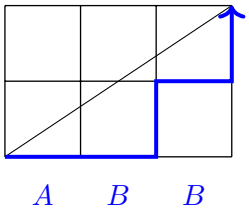
Remark 2.2.12. The uniqueness conjecture of Frobenius [Fro13] is equivalent to saying that the parametrization map $t \in (0,1) \cap \mathbb{Q} \mapsto m^t$ is injective.

2.2.2 Christoffel words

Together with Cohn matrices that we will describe later, Chritoffel words provides a combinatorial tool to compute the function $t \mapsto m^t$ without exploring the Markov and Stern-Brocot trees.

Definition 2.2.13. Let $t = \frac{r}{s} \in (0,1)$ be a rational number. Consider a line of slope t in the lattice $\mathbb{Z}^2$, and construct the lattice path closest to this line, under this line, from the origin $(0,0)$ to the point (s,r). Encode this path by denoting A for a horiztonal step and B for a hook composed with a horizontal step and a vertical step.
The *Christoffel word* w^t of slope t is the binary word over $\{A,B\}$ obtained in this way.
To be complete, we also put $w^{\frac{0}{1}} = A$ and $w^{\frac{1}{1}} = B$.

Example 2.2.14. Consider $t = \frac{2}{3}$. The corresponding Christoffel word is $w^{\frac{2}{3}} = ABB$.



Definition 2.2.15. The *Christoffel tree* is the infinite binary tree rooted at (A, AB, B) and such that the two children of a node labeled (u,v,w) have labels (u, uv, v) (to the left) and (v, vw, w) (to the right).

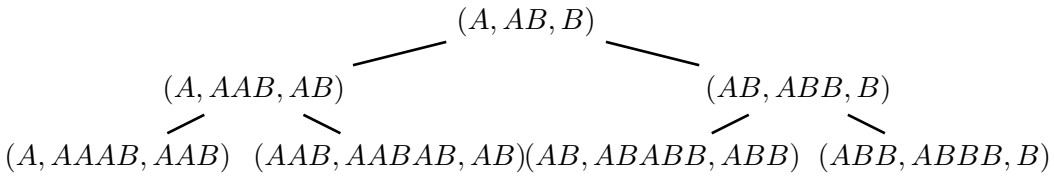


Figure 2.3: First levels of the Christoffel tree

The following result relates Christoffel words and the Stern-Brocot tree. A proof is given in [Aig13, Theorem 7.6].

Proposition 2.2.16. *The Christoffel tree contains all the Christoffel words and the parametrization $t \mapsto w^t$ by rational numbers between 0 and 1 is compatible with the bijection between the Christoffel tree and the Stern-Brocot tree.*
In other words, let $t \in (0,1)$ be a rational number and w^t the associated Christoffel word. Then the node in the Stern-Brocot tree whose middle coordinate is t is at the same place as the node in the Christoffel tree whose middle coordinate is w^t.

Example 2.2.17. The Fibonacci branch corresponds, in the Christoffel tree, to the words with only one B at the end.

$$w^{\frac{1}{1}} = B,\ m^{\frac{1}{2}} = AB,\ m^{\frac{1}{3}} = AAB,\ m^{\frac{1}{4}} = AAAB,\ m^{\frac{1}{5}} = AAAAB, \cdots$$

Example 2.2.18. The Pell branch corresponds, in the Christoffel tree, to the words with only one A at the beginning.

$$w^{\frac{1}{2}} = AB,\ m^{\frac{2}{3}} = ABB,\ m^{\frac{3}{4}} = ABBB,\ m^{\frac{4}{5}} = ABBBB,\ m^{\frac{5}{6}} = ABBBBB, \cdots$$

The shape of Christoffel words is constraint.

Proposition 2.2.19 ([Aig13])**.** *The Christoffel words are of the form $w^t = A\widehat{w}^t B$ where $\widehat{w}^t$ is a palindrome, except $w^{\frac{0}{1}} = A$ and $w^{\frac{1}{1}} = B$.*

2.2.3 Cohn matrices

The Cohn matrices, defined by Cohn in [Coh55], translate computations with Christoffel words and Markov numbers in terms of product of matrices in $\mathrm{SL}_2(\mathbb{Z})$. It provides a very convenient tool to handle big Markov numbers.

Definition 2.2.20. Let $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. It is called a Cohn matrix when b is a Markov number and $b = \frac{\mathrm{Tr}(C)}{3}$. In this case, we also say that C is a Cohn matrix associated to the Markov number b.

Example 2.2.21. The matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is a Cohn matrix associated to 1. The matrix $B = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ is a Cohn matrix associated to 2. Notice that these matrices can be expressed simply in terms of the usual generators of the modular group, T and L (see (1.4),(1.5) in Chapter 1).

$$A = TL \text{ and } B = T^2L^2.$$

Aigner classsifies Cohn matrices by grouping them in infinite families, parametrized by integers.

Definition 2.2.22. Let $n \in \mathbb{Z}$. Define

$$A(n) = \begin{pmatrix} n & 1 \\ 3n - n^2 - 1 & 3 - n \end{pmatrix} \text{ and } B(n) = \begin{pmatrix} 2n+1 & 2 \\ 4n - 2n^2 + 2 & 5 - 2n \end{pmatrix}.$$

Then for any rational number $t \in (0,1)$, define $C^t(n)$ to be the image of the Christoffel word w^t by the monoïd map $\{A, B\}^\star \longrightarrow \mathrm{SL}_2(\mathbb{Z})$ sending A to $A(n)$ and B to $B(n)$.
For the sake of completeness, we also denote $C^0(n) := A(n)$ and $C^1(n) := B(n)$.

Proposition 2.2.23 ([Aig13])**.** *For any $n \in \mathbb{Z}$ and any $t \in [0,1] \cap \mathbb{Q}$, the matrix $C^t(n)$ is a Cohn matrix associated to m^t. Moreover, any Cohn matrix associated to m^t is among the family $\{C^t(n), n \in \mathbb{Z}\}$.*

Definition 2.2.24. Let $n \in \mathbb{Z}$. The nth *Cohn tree* is an infinite binary tree rooted at $(A(n), A(n)B(n), B(n))$ and such that the two children of a node labeled with (M_1, M_2, M_3) have labels (M_1, M_1M_2, M_2) (to the left) and (M_2, M_2M_3, M_3) (to the right).

Example 2.2.25. For $n = 1$, we have $A(1) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, and $B(1) = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$.
The choice $n = 2$ is the most common in the literature about Markov numbers, it corresponds to matrices in Example 2.2.21.
For $n = 3$, we have $A(3) = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$, and $B(3) = \begin{pmatrix} 7 & 2 \\ -4 & -1 \end{pmatrix}$.

Let us summarize the bijections we introduced.

$$\begin{array}{ccc} \text{Rational triples} & & \text{Christoffel words} \\ (s,t,u) & \longleftrightarrow & (w^s, w^t, w^u) \\ t = s \oplus_F u & & w^t = w^s w^u \\ \updownarrow & & \updownarrow \\ \text{Cohn matrices} & & \text{Markov triples} \\ (C^s, C^t, C^u) & \longleftrightarrow & (m^s, m^t, m^u) \\ C^t = C^s C^u & & (m^s)^2 + (m^t)^2 + (m^u)^2 = 3m^s m^t m^u. \end{array} \tag{2.3}$$

2.3 A deformation of Markov numbers

Because traces of q-matrices have interesting combinatorial properties, it is natural to define a deformation of Markov numbers as traces of q-Cohn matrices. This idea was investigated by Kogiso [Kog20] and Kantarcı Oğuz [KO25] with a particular case of Cohn matrices. Recently, with Evans, Morier-Genoud and Ovsienko, we worked on a more global setting, providing uniqueness results and an alternative combinatorial model describing q-Markov numbers [EJMGO26]. This section is based on this paper.

2.3.1 q-Markov numbers as traces of q-Cohn matrices

Recall that Cohn matrices are classified in families depending on a parameter $n \in \mathbb{Z}$, as in Definition 2.2.23, and can be obtained by products of two elementary matrices $A(n)$

and $B(n)$. We compute their q-versions $A(n)_q$ and $B(n)_q$ in $\mathrm{PSL}_{2,q}(\mathbb{Z})$. In order to fix a representant in $\mathrm{GL}_2(\mathbb{Z}[q,q^{-1}])$ (the elements of $\mathrm{PSL}_{2,q}(\mathbb{Z})$ are only defined up to a signed power of q), we impose the condition

$$\det(A(n)_q) = \det(B(n)_q) = 1.$$

Proposition 2.3.1 ([EJMGO26])**.** *The q-deformation of the matrix $A(n)$ is as follows:*

$$A(n)_q = \begin{pmatrix} q^{2-n}[n]_q & q^{1-n} \\ [n]_q[3-n]_q - q^{n-1} & q^{-1}[3-n]_q \end{pmatrix}. \tag{2.4}$$

And the q-deformation of the matrix $B(n)$ is:

$$B(n)_q = \begin{pmatrix} q^{1-n}[n+1]_q[2]_q - q & q^{-n}[2]_q \\ [2]_q((q+q^{-1})[n]_q - q^{2-n}[n+1]_q[n-1]_q) & q^{-2}[3-n]_q[2]_q - q^{-1} \end{pmatrix}. \tag{2.5}$$

Proof. First check that $A(n) = L^{3-n}\, S\, L^n$, where L and S are defined in (1.4),(1.5). The q-deformation of $A(n)$ is then given by

$$L_q^{3-n}\, S_q\, L_q^n.$$

One has

$$\begin{aligned} L_q^{3-n}\, S_q\, L_q^n &= \begin{pmatrix} q^{3-n} & 0 \\ q[3-n]_q & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \begin{pmatrix} q^n & 0 \\ q[n]_q & 1 \end{pmatrix} \\ &= \begin{pmatrix} q^{4-n}[n]_q & q^{3-n} \\ q^2[n]_q[3-n]_q - q^{n+1} & q[3-n]_q \end{pmatrix}. \end{aligned}$$

and we divide through by q^2 so that the determinant is 1. Finally one gets $A(n)_q = q^{-2}L_q^{3-n}\, S_q\, L_q^n$.

Then $B(n)$ can be expressed in terms of $A(n)$. Indeed, $B(n) = A(n)A(n+1)$, and therefore

$$B(n)_q = A(n)_q A(n+1)_q.$$

After a simple computation, one deduces the formula for $B(n)_q$. □

Example 2.3.2. The following special choices $n = 0$, $n = 1$ and $n = 2$ are particularly useful in practice.

$$A(0)_q = \begin{pmatrix} 0 & q \\ -q^{-1} & q^{-1}[3]_q \end{pmatrix}, \quad B(0)_q = \begin{pmatrix} q^2 & [2]_q \\ q[2]_q & q^{-2}[4]_q + 1 \end{pmatrix},$$

$$A(1)_q = \begin{pmatrix} q & 1 \\ q & q^{-1}[2]_q \end{pmatrix}, \quad B(1)_q = \begin{pmatrix} [3]_q & q^{-1}[2]_q \\ q^{-1}[4]_q & q^{-2}[3]_q \end{pmatrix},$$

$$A(2)_q = \begin{pmatrix} [2]_q & q^{-1} \\ 1 & q^{-1} \end{pmatrix}, \quad B(2)_q = \begin{pmatrix} q^{-1}[4]_q + 1 & q^{-2}[2]_q \\ q^{-1}[2]_q & q^{-2} \end{pmatrix}$$

From $A(n)_q$ and $B(n)_q$, one defines the deformation of the whole nth Cohn tree. For $t \in (0,1) \cap \mathbb{Q}$, we denote by $C^t(n)_q$ the q-deformation of the Cohn matrix associated to the Markov number m^t in the nth Cohn tree. Conveniently, we prove that the trace of $C^t(n)_q$ does not depend on the parameter n.

Proposition 2.3.3 ([EJMGO26]). *Let $t \in [0,1]$. Then for any $n \in \mathbb{Z}$, $\mathrm{Tr}(C^t(n)_q)$ does not depend on n and is a Laurent polynomial divisible by $[3]_q$.*

Proof. Let us proceed by induction in the nth Cohn tree. By formulas of $A(n)_q$ and $B(n)_q$ given in Proposition 2.3.1, we get $\mathrm{Tr}(A(n)_q) = q^{-1}[3]_q$, $\mathrm{Tr}(B(n)_q) = q^{-1}[3]_q(q+q^{-1})$ and $\mathrm{Tr}(A(n)_q B(n)_q) = q^{-3}[3]_q[5]_q$.
Let (M, MN, N) be a triple in the nth q-deformed Cohn tree. Either it comes from a right-going edge in the Cohn tree, either from a left-going edge. In the first case, the parent is (MN^{-1}, M, N) and in the second case it is $(M, N, M^{-1}N)$.
Suppose that $\mathrm{Tr}(M)$ and $\mathrm{Tr}(N)$ do not depend on n and are divisible by $[3]_q$, and according to the relevant case suppose that $\mathrm{Tr}(MN^{-1})$ or $\mathrm{Tr}(M^{-1}N)$ is independant of n and divisible by $[3]_q$. These properties propagate to the matrix MN thanks to the well-known formulas for 2×2 matrices

$$\mathrm{Tr}(MN) = \mathrm{Tr}(M)\,\mathrm{Tr}(N) - \mathrm{Tr}(MN^{-1}) = \mathrm{Tr}(M)\,\mathrm{Tr}(N) - \mathrm{Tr}(M^{-1}N). \tag{2.6}$$

□

Definition 2.3.4. Let $t \in [0,1] \cap \mathbb{Q}$ and let m^t be the corresponding Markov number. We define the q-deformation of m^t from the trace of any q-Cohn matrix C_q^t associated to m^t:

$$m_q^t := \frac{\mathrm{Tr}(C_q^t)}{q^{-1}[3]_q}.$$

Remark 2.3.5. We choose to divide the trace of the q-Cohn matrix by $q^{-1}[3]_q$ instead of $[3]_q$ to get more symmetric polynomials.

Example 2.3.6. The first q-deformations of the Markov numbers are as follows

$$\begin{aligned}
1_q &= 1,\\
2_q &= q + q^{-1},\\
5_q &= q^2 + q + 1 + q^{-1} + q^{-2},\\
13_q &= q^3 + 2q^2 + 2q + 3 + 2q^{-1} + 2q^{-2} + q^{-3},\\
29_q &= q^4 + 2q^3 + 4q^2 + 5q + 5 + 5q^{-1} + 4q^{-2} + 2q^{-3} + q^{-4},\\
34_q &= q^4 + 3q^3 + 4q^2 + 6q + 6 + 6q^{-1} + 4q^{-2} + 3q^{-3} + q^{-4},\\
89_q &= q^5 + 4q^4 + 7q^3 + 11q^2 + 14q + 15 + 14q^{-1} + 11q^{-2} + 7q^{-3} + 4q^{-4} + q^{-5},\\
\dots &\quad \dots
\end{aligned}$$

The nice properties satisfied by traces of q-matrices stated in the previous section do not apply directly to q-Markov numbers, because of the division by $q^{-1}[3]_q$ in the definition of m_q^t. However, Kantarcı Oğuz shows that q-deformed Markov numbers can be written as traces of some q-matrices, other than the Cohn matrices. As a byproduct, she deduces the following.

Proposition 2.3.7 (Theorem 7.3 in [KO25]). *Let m_q^t be a q-deformed Markov number. Then*

(i) *the coefficients of* m_q^t *are non-negative integers;*

(ii) m_q^t *is monic, i.e. its leading order coefficient equals* 1*;*

(iii) m_q^t *is symmetric i.e.*

$$m_{q^{-1}}^t = m_q^t;$$

(iv) m_q^t *is unimodal except* $m_q^1 = q + q^{-1}$.

We will give an other combinatorial proof of these properties (except unimodality).

Remark 2.3.8. We distinguish between palindromic polynomials and symmetric Laurent polynomials. Recall that we defined a palindromic polynomial to be an element $f \in \mathbb{Z}[q]$ such that $q^{\deg(f)} f(q^{-1}) = f(q)$, and we define a symmetric Laurent polynomial to be an element $g \in \mathbb{Z}[q, q^{-1}]$ such that $g(q^{-1}) = g(q)$.
This is almost the same notion, as any symmetric Laurent polynomial can be converted into a palindromic polynomial, but the converse is not true. For example take $f(q) = 1+q$, which is palindromic but can not be turned into a symmetric Laurent polynomial.

2.3.2 A q-deformed Markov equation

Consider the following q-deformation of the Markov equation (2.2) .

$$x^2 + y^2 + z^2 \quad = \quad q^{-1} [3]_q \, xyz \; + \; (q-1)(q^{-1} - 1). \tag{2.7}$$

Definition 2.3.9. We call *solution* of (2.7) a triple (a_q, b_q, c_q) of Laurent polynomials with integer coefficients, such that the specialization at $q = 1$ is a classical Markov triple (a, b, c).

Theorem 2.3.10 ([EJMGO26])**.** *Let* (s, t, u) *be a Farey triple. The triple* (m_q^s, m_q^t, m_q^u) *is the unique solution of* (2.7) *which specializes to* (m^s, m^t, m^u) *at* $q = 1$.

The proof of Theorem 2.3.10 is based on the following recurrent procedure that provides one way to calculate the q-Markov numbers explicitly. This recurrence is very similar to the classical case.

Proposition 2.3.11 ([EJMGO26])**.** *Every solution of Equation* (2.7) (a_q, b_q, c_q)*, in the sense of Definition 2.3.9, can be obtained from the triple* $(1, 1, 1)$ *by performing a sequence of transformations* $(a_q, b_q, c_q) \mapsto (a_q, b_q, c_q')$*, where*

$$c_q' \quad := \quad q^{-1} [3]_q \, a_q b_q - c_q, \tag{2.8}$$

combined with permutations of $a_q, b_q,$ *and* c_q.

Proof. It is a simple direct computation to check that if (a_q, b_q, c_q) satisfies (2.7), then (a_q, b_q, c_q') also satisfies (2.7), and c_q' has integer coefficients. As in the classical Markov case, this is just a reformulation of the Vieta formulas for a quadratic equation. In addition, evaluating at $q = 1$, one obtains

$$c' = 3ab - c = \frac{a^2 + b^2}{c} > 0,$$

so that $(a_q, b_q, c'_q)|_{q=1} = (a, b, c')$ is a Markov triple.

We want to show that performing a sequence of transformations of the type (2.8) on an arbitrary solution (a_q, b_q, c_q) of (2.7) one can eventually arrive at the triple $(1, 1, 1)$.

Denote by d_1, d_2, d_3 the degrees of the polynomials a_q, b_q, c_q respectively, and by $\alpha_1, \alpha_2, \alpha_3$ the leading coefficients of the polynomials a_q, b_q, c_q respectively. We assume $d_1 \leq d_2 \leq d_3$ and proceed by induction on d_3.

Case $d_1 = d_2 = d_3 = 0$. In this case $(a_q, b_q, c_q) = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$ is a Markov triple satisfying in addition the relation

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = (q + 1 + q^{-1})\, \alpha_1\alpha_2\alpha_3 + 2 - q - q^{-1}.$$

Looking at the coefficients of q and q^{-1}, this implies $\alpha_1\alpha_2\alpha_3 = 1$. The only classical Markov triple satisfying this is $(1, 1, 1)$, so $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

Case $d_3 \geq 1$. Since (a_q, b_q, c_q) satisfies (2.7), one obtains the following relations on the degrees and leading coefficients:

$$2d_3 = d_1 + d_2 + d_3 + 1, \qquad \alpha_3^2 = \alpha_1\alpha_2\alpha_3.$$

In the expression of c'_q the term $q^{-1}\,[3]_q\, a_q b_q$ has degree $d_1 + d_2 + 1$ and leading coefficient $\alpha_1\alpha_2$. The above relations imply that $q^{-1}\,[3]_q\, a_q b_q$ has degree d_3 and the leading coefficient α_3, the same as c_q. Hence as the higher order term in $c'_q = q^{-1}\,[3]_q\, a_q b_q - c_q$ vanishes, and therefore c'_q is of degree d'_3, which is strictly less than d_3.

Repeating successively transformations on the polynomial of the triple with greatest degree will eventually lead to a triple of polynomials of degree 0, which as seen above can only be $(1, 1, 1)$. Observing that the transformation (2.8) is an involution one gets Proposition 2.3.11. □

Remark 2.3.12. The symmetry property of q-Markov numbers can be obtained inductively using the q-mutation rule (2.8) and the fact that $(1, 1, 1)$ is a triple of symmetric Laurent polynomials.

Proof of Theorem 2.3.10

Proposition 2.3.11 together with the fact that a Markov triple appears exactly once in the Markov tree imply that there is a unique solution of Equation (2.7) for each Markov triple (m^s, m^t, m^u). To conclude the proof of Theorem 2.3.10, it is enough to check that the triple (m^s_q, m^t_q, m^u_q) is indeed a solution of Equation (2.7). This is done by induction on the Markov tree.

The first q-triple $(1, q^{-2}[5]_q, q + q^{-1})$ is a solution of (2.7).
For the induction step, remember the trace formulas for 2×2 matrices used in the proof of Proposition 2.3.3:

$$\mathrm{Tr}(MN) = \mathrm{Tr}(M)\,\mathrm{Tr}(N) - \mathrm{Tr}(MN^{-1}) = \mathrm{Tr}(M)\,\mathrm{Tr}(N) - \mathrm{Tr}(M^{-1}N).$$

This means that $m_q^{s\oplus t} = q^{-1}[3]_q m_q^t m_q^s - m_q^u$, which coincides with the q-mutation rule (2.8), so the triple $(m_q^s, m_q^{s\oplus t}, m_q^t)$ is a solution provided that (m_q^s, m_q^t, m_q^u) is a solution itself. Similarly, $(m_q^t, m_q^{t\oplus u}, m_q^u)$ is a solution of (2.7), and this finishes the induction step and the proof of Theorem 2.3.10.

Entries of the q-Cohn matrices

We conclude by a relationship between traces of q-Cohn matrices and the upper-right coefficient. We will use this result in the next section.

Lemma 2.3.13. *Let $C_q^t = (c_{ij})$ be a q-Cohn matrix, associated to the q-Markov number m_q^t. Then*

$$m_q^t = \left(q^{-1} - q^{-2}\right) c_{11} + q^{-1} c_{12} = q^2 c_{12} + q(1-q)c_{22}. \tag{2.9}$$

Proof. Let us give the details for the first equality as the other one is similar. We proceed by induction on the Cohn tree.

• The induction basis. A simple direct computation shows that the relation (2.9) holds for the Cohn matrices $A(n)_q$ and $B(n)_q$ in the initial triple.

• The induction step. Let (M, MN, N) be a triple of Cohn matrices in the Cohn tree. We assume that its parent is $(M, N, M^{-1}N)$, the other case is similar. Suppose the relation (2.9) holds for $C = M, N$, and $M^{-1}N$. We want to show that the relation holds also for $C = MN$.
Consider the coefficients of the matrices

$$M = (m_{ij}), \quad N = (n_{ij}), \quad P = M^{-1}N = (p_{ij}), \quad C = MN = (c_{ij}),$$

and for simplicity set

$$\alpha := q^{-1} - q^{-2}, \qquad \beta := q^{-1}.$$

The proof actually does not depend on the choice of constants α and β. The relation (2.6) reads

$$\begin{aligned}\frac{\mathrm{Tr}(MN)}{q^{-1}[3]_q} &= q^{-1}[3]_q \left(\frac{\mathrm{Tr}(M)}{q^{-1}[3]_q}\frac{\mathrm{Tr}(N)}{q^{-1}[3]_q}\right) - \frac{\mathrm{Tr}(P)}{q^{-1}[3]_q} \\ &= (m_{11}+m_{22})(\alpha n_{11} + \beta n_{12}) - (\alpha p_{11} + \beta p_{12}),\end{aligned}$$

with $P = M^{-1}N$ so

$$p_{11} = m_{22}n_{11} - m_{12}n_{21} \qquad \text{and} \qquad p_{12} = m_{22}n_{12} - m_{12}n_{22}.$$

One then computes for $C = MN$:

$$\begin{aligned}\frac{\mathrm{Tr}(C)}{q^{-1}[3]_q} &= (m_{11}+m_{22})(\alpha n_{11} + \beta n_{12}) - (\alpha(m_{22}n_{11} - m_{12}n_{21}) + \beta(m_{22}n_{12} - m_{12}n_{22})), \\ &= \alpha(m_{11}n_{11} + m_{12}n_{21}) + \beta(m_{11}n_{12} + m_{12}n_{22}) \\ &= \alpha c_{11} + \beta c_{12}.\end{aligned}$$

Hence the lemma. □

2.3.3 Markov numbers in snake graphs

We begin by recalling the classical construction of the snake graph associated to each Markov number. This construction appears in Harvey Cohn's paper [Coh55, Fig 1.] and in Propp's paper [Pro20]. In the next section, we will add weights to the picture and relate it to our q-Markov numbers.

Choosing a rational number $t = \frac{r}{s} \in [0, 1]$, draw in a $\mathbb{Z}^2$-grid a line of slope t and pick all of the 1×1 squares crossed by this slope, inside the $r \times s$ rectangle. The resulting snake graph is called in [Coh55] the "polygonal approximation" of the diagonal of the $r \times s$ rectangle. We reproduce (see Fig. 2.4 left) a somewhat transposed figure from Cohn's paper (see Fig. 1 in [Coh55]) for $t = \frac{3}{5}$.

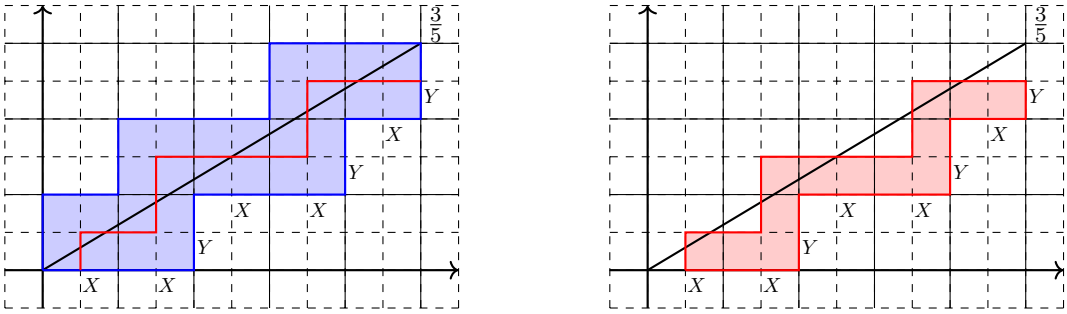


Figure 2.4: Snake graphs. The Cohn snake (left) *vs* the domino snake $\mathcal{G}_t$ (right).

Definition 2.3.14. Let $t = \frac{r}{s} \in (0, 1) \cap \mathbb{Q}$. In the $\mathbb{Z}^2$-grid, consider the polygonal approximation of the diagonal of the $r \times s$ rectangle. Trace the "median zig-zag line" joining the points $(\frac{1}{2}, 0)$ and $(s, r - \frac{1}{2})$ (see the red line on Fig. 2.4, left) and collect the $\frac{1}{2} \times \frac{1}{2}$ square boxes under the line to define the snake graph $\mathcal{G}^t$ associated to the Markov number m^t. To be complete, we also define $\mathcal{G}^{\frac{0}{1}} = \emptyset$ and $\mathcal{G}^{\frac{1}{1}} = \square$.

The snake graph $\mathcal{G}^t$ is called the "domino graph" in [Aig13].

Definition 2.3.15. A *perfect matching* of a graph is a subset of its edges such that every vertex of the graph is incident to exactly one edge of the matching.

Perfect matchings in simplest snake graphs are collected in Fig. 2.5.

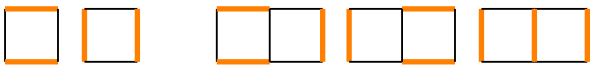


Figure 2.5: Examples of perfect matchings. There are two perfect matchings for a single box and three for a two-box snake.

Alternatively, the graph $\mathcal{G}^t$ can be constructed directly from the Christoffel words. This will require the notion of the dual of a snake graph, introduced in [Pro20], see also [Cla20]. Recall from Chapter 1, Definition 1.4.47, that a snake graph can be encoded by a continued fraction, giving the sequence of lenghts in its upper boundary.

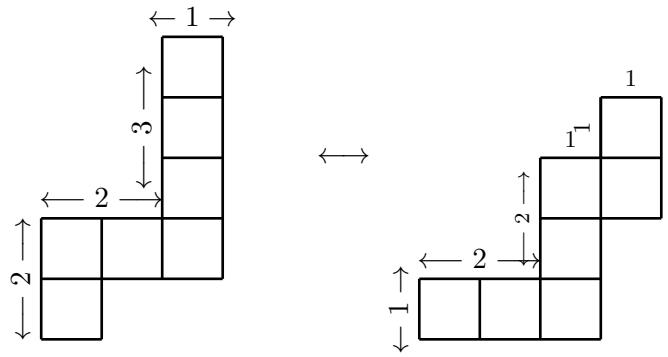
Definition 2.3.16. Let $\mathcal{G}$ be a snake graph, encoded by the sequence $(a_1, \cdots, a_{2m})$. The dual sequence $(a_1, \cdots, a_{2m})^\star$ is defined by the following rules:

- replace each a_i by a sum $1+1+\cdots+1$ adding up to a_i,
- exchange formally each symbol $'+'$ by a comma and each comma by a $'+'$,
- compute each new sum between the commas,
- if necessary, change the last coefficient $b \leftrightarrow b-1,1$ so that the lenght of the sequence is even.

The dual of the snake graph $\mathcal{G}$, denoted by $\mathcal{G}^\star$, is the snake graph encoded by $[a_1, \cdots, a_{2m}]^\star$.

The involution mapping a continued fraction expansion to its dual was studied by Uludağ and Ayral under the name of Jimm involution [UA15].

Example 2.3.17. The dual sequence of $(2,2,3,1)$ is $(1,1+1,1+1,1,1,1) = (1,2,2,1,1,1)$. In terms of snake graphs, this means that the dual of $\mathcal{G}_{\frac{22}{9}}$ is $\mathcal{G}_{\frac{27}{19}}$.



One of the interest of the dual construction is that it provides an explicit bijection between the lattice paths of a snake graph $\mathcal{G}$ (see Definition 1.4.49) and the perfect matchings of the dual $\mathcal{G}^\star$.

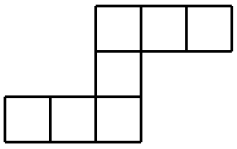
Proposition 2.3.18 ([Pro20, Cla20])**.** *Let $\mathcal{G}$ be a snake graph. There is a distributive lattice structure on the set of perfect matchings of $\mathcal{G}$, and it is isomorphic to the distributive lattice of lattice paths in $\mathcal{G}^\star$. Therefore, any perfect matching of $\mathcal{G}$ corresponds to a lattice path in $\mathcal{G}^\star$, and the rank of the perfect matching is the area of the lattice path.*

Let us come back to the snake graphs associated to Markov numbers. For $t \in (0,1) \cap \mathbb{Q}$, let w^t the corresponding Christoffel word. Associate to w^t a continued fraction as follows:

- replace each A by $(1,1)$,
- replace each B by $(2,2)$,
- except the first A and the last B which are each replaced by 2.

Denote by x^t the rational number computed this way. With notations of Chapter 1, Definition 1.4.47, the snake graph $\mathcal{G}^t$ is then the dual of the snake graph associated to x^t, $\mathcal{G}^t = \mathcal{G}^\star_{x^t}$.

Example 2.3.19. Let us detail the case of the Markov number $m^{\frac{2}{3}} = 29$. Its Christoffel word is AB^2 so the associated continued fraction is $[2,2,2,2]$, and its dual is $[2,2,2,2]^\star = [1,2,2,3]$. The snake graph $\mathcal{G}^{\frac{2}{3}}$ is then

$\mathcal{G}^{\frac{2}{3}} = \mathcal{G}^{\star}_{\frac{29}{12}} =$ 

Notice that the numerator of the rational number $x^t = [2, 2, 2, 2]$ is precisely the Markov number m^t. This is not a coincidence.

The following statement is a well-known fundamental result explaining the combinatorial nature of Markov numbers. A proof can be found in Aigner's book [Aig13].

Theorem 2.3.20. *Every Markov number m^t is equal to the number of perfect matchings in $\mathcal{G}^t$.*

Equivalently, m^t is the number of lattice paths in the dual snake graph $\mathcal{G}_{x^t}$. This result first appears in [Pro20, Thm. 7.1]. A self-contained transparent proof can be found in [Aig13, Thm. 7.12]. The statement can be deduced from the result of [CS20] about continued fractions. Note that the result of [CS20] is actually more general. Another approach based on connections with cluster algebras is outlined in [Sch24].

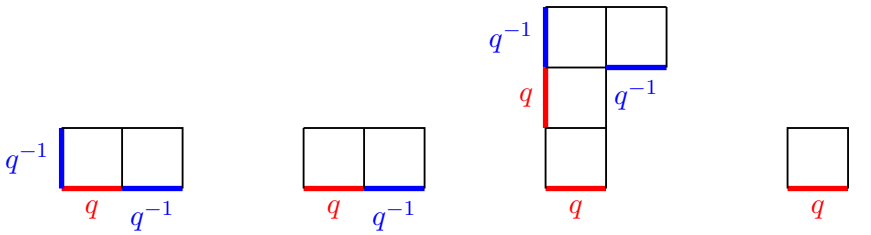
2.3.4 Weighted perfect matchings

The main goal of this section is to give a generalization of Theorem 2.3.20. To this end we will assign weights to the edges of $\mathcal{G}^t$.

Definition 2.3.21. The *weighted snake graph* $\mathcal{G}^t(q)$ is obtained from the snake graph $\mathcal{G}^t$ by assigning weights on the edges of $\mathcal{G}^t$ according to the following rule.

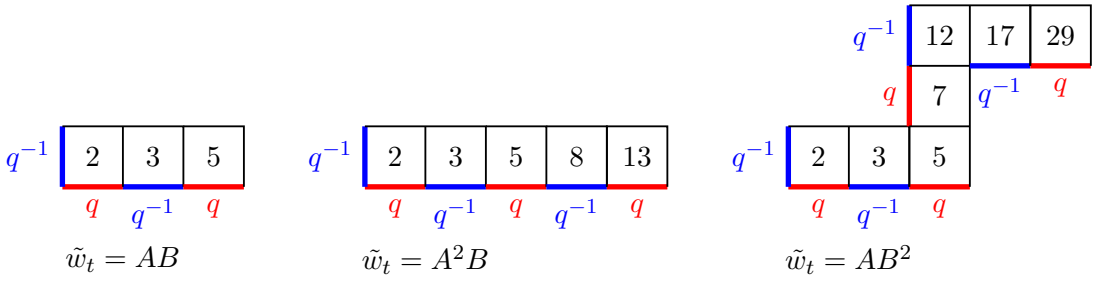
1. Vertical edges on the Western boundary of $\mathcal{G}^t$ are assigned with weights $q^{-1}, q, \ldots$ alternating from bottom to top.
2. Horizontal edges on the Southern boundary of $\mathcal{G}^t$ are assigned with weights $q, q^{-1}, \ldots$ alternating from left to right.
3. Other edges of $\mathcal{G}^t$ are assigned with weight 1.

More precisely we consider the Christoffel word w^t. Each letter in w^t gives rise to a piece of snake graph according to the following description:



All the pieces are connected by gluing the rightmost vertical edge of the previous piece to the bottom leftmost vertical edge of the next piece.

Example 2.3.22. (a) Examples of $\mathcal{G}^t(q)$, for $t = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, corresponding to the Markov numbers $5, 13, 29$, respectively.



(b) Figure 2.4 gives an example for $t = \frac{3}{5}$.

Definition 2.3.23. (i) Given a graph $\mathcal{G}$ with weighted edges, we will use the standard notion of the *weighted perfect matching* in $\mathcal{G}$. For a given perfect matching $\mathfrak{m}$, we define its weight, $\mathrm{wt}(\mathfrak{m})$, as the product of the weights of its edges. The weight is a power of q.

(ii) Given a snake graph $\mathcal{G}^t(q)$ with weighted edges, we define the *weighted number of perfect matchings* as the generating function for the weighted perfect matchings:

$$\mu^t(q) := \sum_{\mathfrak{m}} \mathrm{wt}(\mathfrak{m}).$$

Other terms sometimes used in the dimer theory are "statistics" or "statistical sum".

We give a q-deformed version of Theorem 2.3.20 relating Markov numbers and perfect matchings in snake graphs.

Theorem 2.3.24 ([EJMGO26]). *The q-deformed Markov number m_q^t is equal to the weighted number of perfect matchings $\mu^t(q)$ in the snake graph $\mathcal{G}^t(q)$.*

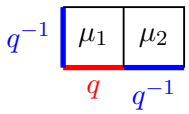
Our proof will follow that of Theorem 7.12 of [Aig13].

Proof. Fix a rational $t \in (0,1)$ and the corresponding Christoffel word

$$w^t = Aw_1 \cdots w_s B,$$

with letters $w_i \in \{A, B\}$. Let n be the number of boxes in the snake graph $\mathcal{G}^t(q)$. We count the boxes travelling along the snake from left to right and bottom to top.

For each $i \in [\![1, n]\!]$, let us denote by μ_i the weighted number of perfect matchings of the partial snake graph containing the first i boxes. It is convenient to label the i-th box with the value μ_i. In particular, for the initial snake graph one has



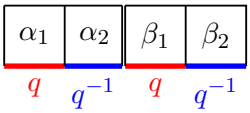
with

$$\mu_1 = q + q^{-1}, \qquad \mu_2 = q + q^{-1} + q^{-2}. \tag{2.10}$$

Here μ_1 is easily obtained by enumerating the 2 perfect matchings on a single box and μ_2 is obtained by enumerating the 3 perfect matchings on the graph with two boxes (see example of Fig. 2.5).

We can compute recursively μ_i in the snake graph $\mathcal{G}^t(q)$, looking at how it evolves through the addition of a type A piece or a type B piece of graph (see [Aig13], p.146).

Attachment of an A-piece



Then

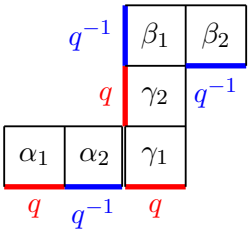
$$\beta_1 = \alpha_2 + q\alpha_1, \qquad \beta_2 = \beta_1 + q^{-1}\alpha_2,$$

which can be rewritten as

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} q & 1 \\ q & 1+q^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Observe that the matrix $\begin{pmatrix} q & 1 \\ q & 1+q^{-1} \end{pmatrix}$ is nothing but the q-deformed Cohn matrix $A(1)_q$, already calculated in Proposition 2.3.1.

Attachment of a B-piece (except for the final one)



One easily obtains the following relations between the weighted number of perfect matchings:

$$\begin{aligned} \gamma_1 &= \alpha_2 + q\alpha_1 \\ \gamma_2 &= \gamma_1 + q^2\alpha_1 \\ \beta_1 &= \gamma_2 + q^{-1}\gamma_1 \\ \beta_2 &= \beta_1 + q^{-2}\gamma_1 \end{aligned}$$

from which one deduces

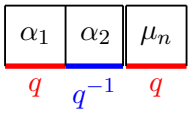
$$\begin{aligned} \beta_1 &= \left(q^2+q+1\right)\alpha_1 + \left(1+q^{-1}\right)\alpha_2 \\ \beta_2 &= \left(q^2+q+1+q^{-1}\right)\alpha_1 + \left(1+q^{-1}+q^{-2}\right)\alpha_2 \end{aligned}$$

which again can be rewritten as

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = B(1)_q \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

where $B(1)_q$ is the q-deformed Cohn Matrix companion to $A(1)_q$.

Attachment of the final B-piece



In this case, we have

$$\mu_n = \begin{pmatrix} q & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

We conclude that the weighted number of perfect matchings of $\mathcal{G}^t(q)$ is given by

$$\mu_n = \begin{pmatrix} q & 1 \end{pmatrix} M_s \cdots M_1 \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},$$

where for $k \in \{1, \cdots, s\}$,

$$M_k = \begin{cases} A(1)_q, & \text{if } w_k = A, \\ B(1)_q, & \text{if } w_k = B. \end{cases}$$

and where μ_1 and μ_2 are given by (2.10).

Observing now that

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = B(1)_q \begin{pmatrix} q^{-1} - q^{-2} \\ q^{-1} \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} q & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} A(1)_q,$$

and using the fact that $w_1 \cdots w_s$ is a palindrome, we get

$$\mu_n = \begin{pmatrix} 1 & 0 \end{pmatrix} A(1)_q \, M_1 \cdots M_s \, B(1)_q \begin{pmatrix} q^{-1} - q^{-2} \\ q^{-1} \end{pmatrix}.$$

The matrix

$$C_q^t = A(1)_q M_1 \cdots M_s B(1)_q =: (c_{ij})$$

is precisely the q-Cohn matrix $C^t(1)_q$ associated to the Markov number m_q^t, therefore

$$\mu_n = \begin{pmatrix} 1 & 0 \end{pmatrix} C_q^t \begin{pmatrix} q^{-1} - q^{-2} \\ q^{-1} \end{pmatrix} = \begin{pmatrix} q^{-1} - q^{-2} & q^{-1} \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = \left(q^{-1} - q^{-2}\right) c_{11} + q^{-1} c_{12}.$$

Applying Lemma 2.3.13, we finally get

$$\mu^t(q) = \mu_n = \frac{\mathrm{Tr}(C_q^t)}{q^{-1}[3]_q} = m_q^t,$$

as desired. □

Remark 2.3.25. The interpretation of q-Markov numbers as weighted number of perfect matchings gives another proof of positivity of the coefficients of these Laurent polynomials. It also justifies that q-Markov numbers are monic, as there is only one perfect matching of minimal weight, choosing all edges of weight q^{-1} and completing with edges of weight 1.

2.3.5 Weighted uniqueness conjecture

As a consequence of the combinatorial interpretation of q-Markov numbers, we get a q - version of the uniqueness conjecture on maxima of Markov triples.

Theorem 2.3.26 ([EJMGO26]). *The q-deformed map $t \in [0,1] \cap \mathbb{Q} \longrightarrow m_q^t$ is injective.*

The proof relies on the explicit computation of the parameter t, given a q-Markov number m_q^t.

Lemma 2.3.27. *Let $m_q^t = q^d + \alpha q^{d-1} + \cdots + \alpha q^{1-d} + q^{-d}$ be a q-Markov number. The corresponding rational parameter is expressed in terms of the degree of m_q^t and its coefficient of the term of degree $d-1$:*

$$t = \frac{d-\alpha}{\alpha+1}.$$

Proof. There exists exactly one perfect matching of $\mathcal{G}^t$ with weight q^{-d}, namely "the minimal matching" consisting of all boundary edges of weight q^{-1} completed (in a unique way) with boundary edges of weight 1. Every letter A in the Christoffel word w^t contributes with q^{-1} (except the first one, which counts for q^{-2}), and every letter B contributes with q^{-2} (except the last one which counts for 1). Thus

$$d = \#A + 2\#B - 1.$$

Besides we know, by palindromicity, that α is also the coefficient of the term of degree $1-d$, hence the number of perfect matching of $\mathcal{G}^t$ of weight q^{1-d}. Such a perfect matching is obtained by taking all edges of weight q^{-1}, but one. This is possible only by removing in the minimal matching a horizontal edge of weight q^{-1} with the opposite boundary edge sharing the same box, and replacing these two by the two internal vertical edges of the box. There are $\#A + \#B - 1$ horizontal edges of weight q^{-1}, therefore $\alpha = \#A + \#B - 1$. Finally one gets

$$t = \frac{\#B}{\#A + \#B} = \frac{d-\alpha}{\alpha+1}.$$

Hence the lemma. □

Example 2.3.28. Let us consider the Markov triple $(1,2,5)$, and its deformation $(1_q, 2_q, 5_q) = (1, q+q^{-1}, q^2+q+1+q^{-1}+q^{-2})$. Then we recover the rational parameter t of 5_q by

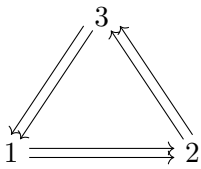
$$\frac{2-1}{1+1} = \frac{1}{2} = t.$$

2.4 Further direction : q-Markov numbers in cluster algebra theory

The theory of cluster algebras by Fomin and Zelevinsky gives an excellent framework to study Markov numbers and the Markov equation. Various work on Markov numbers and (generalized) Markov equation has been produced in this framework, see e.g. [BG25, BBH11, EVW25, LLRS23, Chá12, Pro20].
Here we introduce a 3-parameter deformation of Cohn matrices related to the F-polynomials of the Markov cluster algebra. First we mention some ingredients of the theory of cluster

algebras without going deeply into details. We refer to [FZ07, FWZ24] for an overview and the main definitions.

The *Markov cluster algebra* is built from the following quiver



by applying the standard mutation rules for cluster algebras. In this situation, the mutation rule matches the Vieta jumps defining Markov numbers.

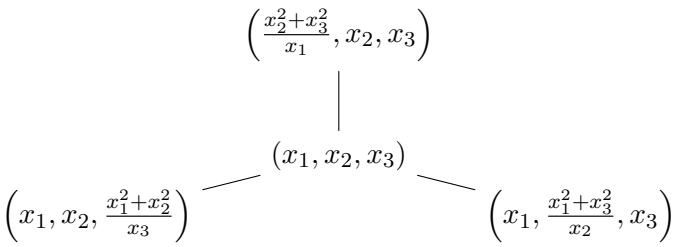


Figure 2.6: Mutation rule for Markov cluster variables

We will hence label the Markov cluster variables with rational numbers and obtain the set $\{\chi_t,\, t \in \mathbb{Q}\}$, so that $\chi_{\frac{0}{1}} = x_1$, $\chi_{\frac{1}{0}} = x_2$, $\chi_{\frac{1}{-1}} = x_3$ are the initial cluster variables.

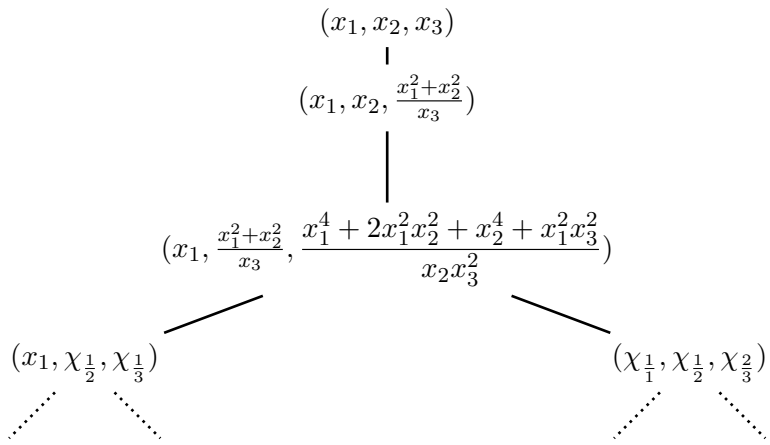


Figure 2.7: Tree of cluster variables in the Markov cluster algebra

In general, cluster variables are elements of the ring $\mathbb{Z}_{>0}[y_1, y_2, y_3][x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$, see [FZ07]. In this setting, one recovers the cluster variables χ_t described above by specializing $y_1 = y_2 = y_3 = 1$, and one obtains the associated F-polynomials F_t as elements of $\mathbb{Z}_{>0}[y_1, y_2, y_3]$ by specializing $x_1 = x_2 = x_3 = 1$.

Specializing all the variables simultaneously to 1, one obtains the Markov numbers

$$m^t = \chi_t(1, 1, 1) = F_t(1, 1, 1).$$

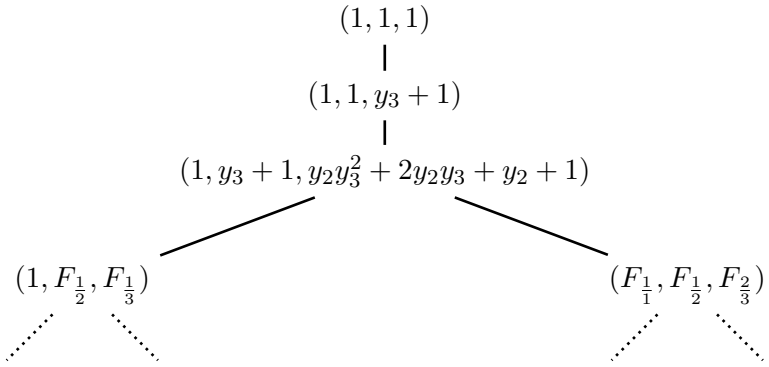


Figure 2.8: Tree of F-polynomials in the Markov cluster algebra

Example 2.4.1. (a) Further examples of Markov cluster variables are as follows.

$$\begin{aligned}
\chi_{\frac{1}{3}} &= \frac{x_1^6+3x_1^4x_2^2+3x_1^2x_2^4+x_2^6+(2x_1^4+2x_1^2x_2^2)x_3^2+x_1^2x_3^4}{x_2^2x_3^3},\\
\chi_{\frac{2}{3}} &= \frac{x_1^8+4x_1^6x_2^2+6x_1^4x_2^4+4x_1^2x_2^6+x_2^8+(2x_1^6+5x_1^4x_2^2+4x_1^2x_2^4+x_2^6)x_3^2+x_1^4x_3^4}{x_1x_2^2x_3^4},\\
\chi_{\frac{3}{4}} &= \frac{x_1^{12}+6x_1^{10}x_2^2+15x_1^8x_2^4+20x_1^6x_2^6+15x_1^4x_2^8+6x_1^2x_2^{10}+x_2^{12}}{x_1^2x_2^3x_3^6}\\
&\quad+\frac{(3x_1^{10}+14x_1^8x_2^2+26x_1^6x_2^4+24x_1^4x_2^6+11x_1^2x_2^8+2x_2^{10})x_3^2}{x_1^2x_2^3x_3^6}\\
&\quad+\frac{(3x_1^8+8x_1^6x_2^2+8x_1^4x_2^4+4x_1^2x_2^6+x_2^8)x_3^4+x_1^6x_3^6}{x_1^2x_2^3x_3^6}.
\end{aligned}$$

(b) The corresponding F-polynomials are the following

$$\begin{aligned}
F_{\frac{1}{3}} &= y_2^2y_3^3+3y_2^2y_3^2+3y_2^2y_3+y_2^2+2y_2y_3+2y_2+1,\\
F_{\frac{2}{3}} &= y_1y_2^2y_3^4+2y_1y_2^2y_3^3+y_2^2y_3^4+y_1y_2^2y_3^2+4y_2^2y_3^3+6y_2^2y_3^2+4y_2^2y_3+2y_2y_3^2+y_2^2\\
&\quad+4y_2y_3+2y_2+1,\\
F_{\frac{3}{4}} &= y_1^2y_2^3y_3^6+2y_1^2y_2^3y_3^5+2y_1y_2^3y_3^6+y_1^2y_2^3y_3^4+8y_1y_2^3y_3^5+y_2^3y_3^6+12y_1y_2^3y_3^4+6y_2^3y_3^5\\
&\quad+8y_1y_2^3y_3^3+2y_1y_2^2y_3^4+15y_2^3y_3^4+2y_1y_2^3y_3^2+4y_1y_2^2y_3^3+20y_2^3y_3^3+3y_2^2y_3^4\\
&\quad+2y_1y_2^2y_3^2+15y_2^3y_3^2+12y_2^2y_3^3+6y_2^3y_3+18y_2^2y_3^2+y_2^3+12y_2^2y_3+3y_2y_3^2\\
&\quad+3y_2^2+6y_2y_3+3y_2+1.
\end{aligned}$$

As recalled in the previous sections the Markov number m^t appears as the top right element of the corresponding Cohn matrix. We introduce a deformation of the Cohn matrices using the parameters (y_1, y_2, y_3) leading to a generalization of this fact. Consider

the 3-parameter quantization governed by

$$\begin{aligned}
\hat{A} &= \begin{pmatrix} y_3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2 & 0 \\ y_2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} y_2 + y_2 y_3 & 1 \\ y_2 & 1 \end{pmatrix}. \\
\hat{B} &= \begin{pmatrix} y_3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2 & 0 \\ y_2 & 1 \end{pmatrix} \begin{pmatrix} y_3 & 0 \\ y_2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} y_1 y_2 y_3^2 + y_2 y_3^2 + 2 y_2 y_3 + y_2 & y_3 + 1 \\ y_2 y_3 + y_2 & 1 \end{pmatrix}.
\end{aligned}$$

Note that these are products of what we can consider as y-deformed versions of two standard generators T and L of $\mathrm{SL}_2(\mathbb{Z})$. Specializing $y_1 = y_2 = y_3 = q$ in $\hat{A}, \hat{B}$ will return the q-deformed Cohn matrices $A(2)_q, B(2)_q$, up to a power of q.

The following statement was a conjecture in [EJMGO26], and a proof was found and communicated to us by Esther Banaian, using tools from her paper with Gyoda [BG25]. Not knowing whether this proof will be part of a future publication or not, we reproduce it here with her consent.

Proposition 2.4.2. *For every rational $0 \leq t \leq 1$, corresponding to the Christoffel word w^t, the deformed matrix $\hat{C}_t$ given by $w^t(\hat{A}, \hat{B})$ has a top right entry equal to the F-polynomial F_t.*

To prove this result, we will work in the model of fence posets instead of snake graphs, to follow the language of Banaian, Gyoda [BG25] and Kantarcı Oğuz, Yıldırım [KOY25]. We will also assign labels $\{1, 2, 3\}$ to the vertices of fence posets, keeping track of which vertex of the Markov quiver they correspond to.

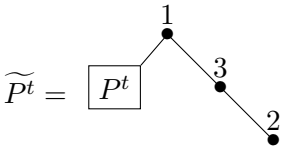
For each rational number $t \in [0, 1]$, recall that we associate a snake graph $\mathcal{G}^t$ whose number of perfect matchings is m^t, see Definition 2.3.14 . Consider the dual of this snake graph, $(\mathcal{G}^t)^\star$, and convert it into a fence poset by replacing each tile by a vertex, and reading the snake graph $(\mathcal{G}^t)^\star$ from bottom to top, for each step up create a downward edge in the fence poset, and for each step to the right create an upward edge in the fence poset. The fence poset obtained in this way is denoted by P^t.

By the bijection between the model of snake graphs and the model of fence posets (see Proposition 2.3.18), the number of ordered ideals in P^t is exactly the Markov number m^t.

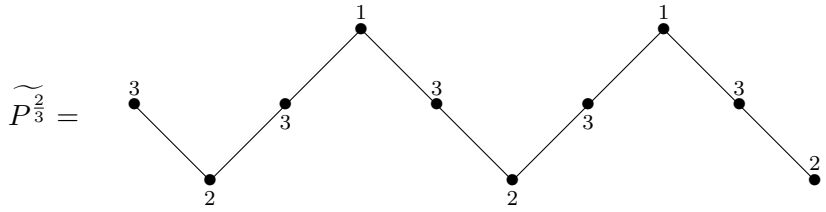
Now, label the vertices of P^t according to the following rules :

- a vertex has label 2 if and only if it is a valley ;
- reading the poset from left to right, the first vertex has label 3 and then alternate one vertex over two with label 3 ;
- the remaining vertices have label 1.

Finally, build from P^t the fence poset $\widetilde{P^t}$ by adding in the end a chain of three vertices going downward, as follows.



Example 2.4.3. For $t = \frac{2}{3}$, we have $m^t = 29$, and the associated fence poset is



We associate to any ordered ideal I in $\widetilde{P^t}$ a label $y(I)$ which is a monomial in y_1, y_2, y_3, such that each vertex $v \in I$ labelled j contributes to y_j, for $j \in \{1, 2, 3\}$. We define four generating functions of ordered ideals in $\widetilde{P^t}$, according to the role played by the first vertex v_{first} of the fence and the last vertex v_{last}.

$$w(\widetilde{P^t}, v_{\text{last}}) := \sum_{\substack{I \text{ ideal} \\ v_{\text{last}} \in I}} y(I),$$

$$w(\widetilde{P^t}, \neg v_{\text{last}}) := \sum_{\substack{I \text{ ideal of } \widetilde{P^t} \\ v_{\text{last}} \notin I}} y(I),$$

$$w(\widetilde{P^t}, v_{\text{last}}, \neg v_{\text{first}}) := \sum_{\substack{I \text{ ideal of } \widetilde{P^t} \\ v_{\text{last}} \in I, v_{\text{first}} \notin I}} y(I),$$

$$w(\widetilde{P^t}, \neg v_{\text{first}}, \neg v_{\text{last}}) := \sum_{\substack{I \text{ ideal of } \widetilde{P^t} \\ v_{\text{first}} \notin I, v_{\text{last}} \notin I}} y(I).$$

Gather these generating functions in a matrix

$$W^t := \begin{pmatrix} w(\widetilde{P^t}, v_{\text{last}}) & w(\widetilde{P^t}, \neg v_{\text{last}}) \\ w(\widetilde{P^t}, v_{\text{last}}, \neg v_{\text{first}}) & w(\widetilde{P^t}, \neg v_{\text{first}}, \neg v_{\text{last}}) \end{pmatrix}.$$

The idea for proving Proposition 2.4.2 is to show that $\hat{C}_t = W^t$, and apply a result of [BG25] ensuring that the top-right entry of W^t is the F-polynomial F_t.

Proof of Proposition 2.4.2. We proceed by induction on the Farey tree to show that for all $t \in [0, 1]$, $W^t = \hat{C}_t$. This equality is clearly true for $\hat{A}$ and $\hat{B}$.
Let (s, t, u) be a Farey triple, with $t = s \oplus_F u$, and suppose that $W^s = \hat{C}_s$ and $W^u = \hat{C}_u$. We want to show that $W^s W^u = W^t$. By Lemma 8.27 in [BG25] (see also Lemma 3.6.7 in Chapter 3), the poset associated to t is given by the concatenation of $\widetilde{P^s}$ and $\widetilde{P^u}$, linking them by an upward edge. In the notations of [KOY25], $\widetilde{P^t} = \widetilde{P^s} \nearrow \widetilde{P^u}$. Applying Theorem 5.2 of [KOY25], we have then $W^t = W^s W^u$.

There only remains to study the top-right entry of W^t. Notice that by construction, the top-right entry of W^t is the generating function of ordered ideals in P^t. By Theorem 8.30 of [BG25], this is exactly the F-polynomial F_t. □

Chapter 3

A geometric perspective on q-numbers

> Je fais, en traversant les groupes et les ronds, sonner les vérités comme des éperons !
>
> *Cyrano de Bergerac*, Edmond Rostand

The strenght of q-real numbers is that they cross different areas of mathematics and can be understood with many points of view. In the previous chapters, our approach was mainly combinatorial. We explore now the geometric side of q-numbers, working in the hyperbolic plane. In Section 3.1, we recall basic knowledge about this topic, for the convenience of the reader. Then we take a closer look to q-Farey operations which were introduced in Chapter 1, and generalize them in Section 3.2. The Section 3.3 explains the main idea of this chapter, which consists of interpreting q-rational numbers as disks in the plane. From this new perspective naturally arise operations on q-rational numbers called the Springborn operations. We describe them in the classical setting in Section 3.4, before studying their q-deformations in Section 3.5. As an illustration of these q-Springborn operations, we define in Section 3.6 a new deformation of Markov numbers, based on the Markov fractions introduced by Springborn.

The material presented here is part of a collaborative work with Olga Paris-Romaskevich and Alexander Thomas, and is available as a preprint [JPRT26].

Main results of Chapter 3. The main motivation and one of the goal of this chapter is creating a bridge between a geometric construction (homothetic centers of a pair of disks) and an algebraic operation (Springborn sum and difference of a pair of rationals). This bridge is made sequentially by

- Theorem 3.5.2, for pairs of rational numbers at Farey distance 1 or 2,
- Theorem 3.5.10, for a larger class of pairs of rational numbers, called the inner-regular and outer-regular pairs.

In the process toward these results, we get other interesting theorems, such as

- Theorem 3.2.7 which generalizes the Farey operations for q-rational numbers,

- Theorem 3.5.5 which give explicit reduced formulas to compute the deformations of Springborn sum and difference of a regular pair of rationals,
- Theorem 3.5.12, providing an explicit formula for the q-deformation of the arithmetic mean of two rational numbers with Farey determinant 1.

Contents

3.1 Framework : hyperbolic geometry

3.1.1 Hyperbolic plane

From now on, denote by $\mathbb{H}$ the hyperbolic plane. We will essentially be working in the half plane model. We recall without proof some classical facts about the geometry of $\mathbb{H}$ that will be needed in the next sections. We refer to [SET09] for an elementary exposition about hyperbolic geometry (in french).

Definition 3.1.1. The *half plane model* realizes $\mathbb{H}$ as the space $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, endowed with the hyperbolic metric $|dz|/\mathrm{Im}(z)$. In this model, the boundary of $\mathbb{H}$ is the real axis completed with ∞. We denote by $\overline{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$ the closure of $\mathbb{H}$.

Definition 3.1.2. An *isometry* of $\mathbb{H}$ is a distance-preserving diffeomorphism of $\mathbb{H}$. We denote by $\mathrm{Isom}(\mathbb{H})$ the group of isometries of $\mathbb{H}$, and $\mathrm{Isom}_+(\mathbb{H})$ the subgroup of orientation-preserving isometries.

Example 3.1.3. The application $z \mapsto -\overline{z}$ is an orientation-reversing isometry of $\mathbb{H}$. For $x \in \mathbb{R}$, the shift $z \mapsto z + x$ is an orientation preserving isometry of $\mathbb{H}$.

Definition 3.1.4. A *Möbius transformation* is an application $\mathbb{C} \longrightarrow \mathbb{C}$ of the form

$$z \mapsto \frac{az+b}{cz+d} \text{ for } a, b, c, d \in \mathbb{R} \text{ such that } ad - bc \neq 0.$$

Remark 3.1.5. The set of Möbius transformations is a group, clearly isomorphic to $\mathrm{PGL}_2(\mathbb{R})$ via fractional linear transformations of matrices (see Definition 1.2.2). However, one can notice that the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ preserves the upper half plane if and only if $ad-bc > 0$. Because of this, we will work with another realization of $\mathrm{PGL}_2(\mathbb{R})$, by assigning to a matrix of determinant -1 the composition of the complex conjugation and the action of the matrix by fractional linear transformation.

$$\begin{array}{ccc} \mathrm{PGL}_2(\mathbb{R}) \times \mathbb{H} & \longrightarrow & \mathbb{H} \\ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z & \longmapsto & \begin{cases} M \cdot z = \frac{az+b}{cz+d} \text{ if } \det(M) > 0 \\ M \cdot \overline{z} = \frac{a\overline{z}+b}{c\overline{z}+d} \text{ if } \det(M) < 0. \end{cases} \end{array} . \tag{3.1}$$

Proposition 3.1.6. *A Möbius transformation preserves the set of Euclidean circles or Euclidean lines in* $\mathbb{C}$.

Definition 3.1.7. Let C be a Euclidean circle in $\mathbb{C}$, of center z_0 and radius r. The *inversion* in C (or with respect to C) is the application $I_C : \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$ sendind a point $z \notin \{z_0, \infty\}$ on the point $I_C(z)$ on the half-line $[z_0, z)$ satisfying

$$|I_C(z) - z_0|\,|z - z_0| = r^2,$$

and exchanging z_0 and ∞.
We extend the definition to inversions with respect to Euclidean lines, which are orthogonal reflections.

Theorem 3.1.8. *The group of orientation-preserving isometries* $\mathrm{Isom}_+(\mathbb{H})$ *of the hyperbolic plane is* $\mathrm{PSL}_2(\mathbb{R})$ *acting by Möbius transformations. The group of isometries* $\mathrm{Isom}(\mathbb{H})$ *is isomorphic to* $\mathrm{PGL}_2(\mathbb{R})$*, acting as* (3.1).

Example 3.1.9. Let C be an Euclidean circle in $\mathbb{C}$ orthogonal to the real axis. The inversion in C defines an orientation-reversing isometry of $\mathbb{H}$. Likewise, the orthogonal reflection with respect to a vertical line defines an orientation-reversing isometry of $\mathbb{H}$. For example, the map $z \mapsto -\overline{z}$ is the orthogonal reflection in the vertical axis of imaginary numbers.

The circles orthgonal to the real axis play an important role in hyperbolic geometry because their upper-parts are geodesics of $\mathbb{H}$.

Definition 3.1.10. A *geodesic* in $\mathbb{H}$ is a curve from the boundary of $\mathbb{H}$ to itself, minimizing the lenght between any pair of its points.

Theorem 3.1.11. *Given two points of $\mathbb{H}$, there is a unique geodesic passing by these two points, and this geodesic is either a vertical half-line, either a half circle orthogonal to the real axis.*

By abuse of language, given a geodesic γ of $\mathbb{H}$, we will say *the inversion in γ* to talk about the inversion with respect to the circle whose upper half is γ. Two properties of geodesics of $\mathbb{H}$ will be of particular use for us.

Proposition 3.1.12. *Let γ_1 and γ_2 be two distinct geodesics of $\mathbb{H}$. Then γ_1 is orthogonal to γ_2 if and only if the inversion in γ_1 preserves γ_2.*

Proposition 3.1.13. *Let γ_1 and γ_2 be two disjoint geodesics of $\mathbb{H}$. There exists a unique geodesic γ_3 orthogonal to both γ_1 and γ_2.*

Remark 3.1.14. If γ_1 and γ_2 are two distinct vertical lines in $\mathbb{H}$, they are not disjoint because they share ∞, so the proposition above does not apply.

3.1.2 Classical Farey tessellation

Recall the Farey graph, constructed in Chapter 1, Definition 1.1.13. It was then viewed as an abstract graph, but we can embed it in $\overline{\mathbb{H}}$, by putting the vertex labelled $x \in \mathbb{Q}_{\geq 0}$ at the point of coordinate $(x, 0)$ in the half plane model. With an axial symmetry, it can be extended to the negative part of the half plane, and this produces an ideal triangulation of $\mathbb{H}$, called the Farey tessellation.

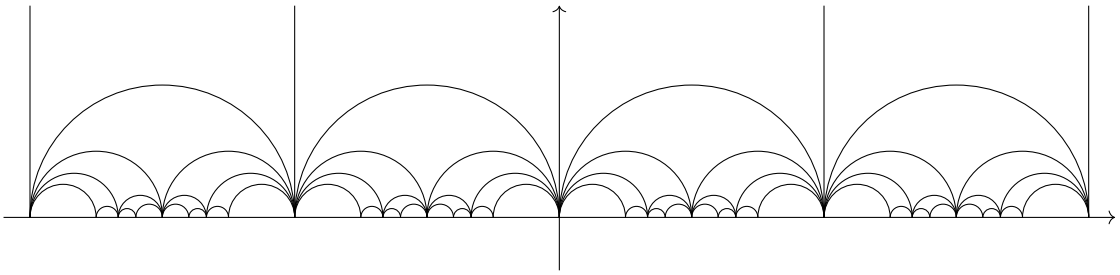


Figure 3.1: The first levels of the Farey tessellation of $\mathbb{H}$, represented in the half plane model.

A symmetry of the Farey tessellation is an isometry of $\mathbb{H}$ preserving the Farey graph. For example, the shift $T : z \mapsto z + 1$ and the reflection $z \mapsto -\overline{z}$ are symmetries of the Farey tessellation. The transformation $z \mapsto -\frac{1}{z}$ is also a symmetry, it is the hyperbolic rotation of π around i. These three transformations generate the group $\mathrm{PGL}_2(\mathbb{Z})$.

Conversely, a symmetry of the Farey tessellation must be an isometry of $\mathbb{H}$ so it is represented by a matrix $M \in \mathrm{GL}_2(\mathbb{R})$ (see Theorem 3.1.8). Since it preserves $\mathbb{P}^1(\mathbb{Q})$, there must be a representant of M with rational coefficients, and even integer coefficients by renormalization. A symmetry of the Farey tessellation must also preserve pairs of Farey determinant 1 (represented by edges of the Farey graph), so it sends the pair $(\frac{1}{0}, \frac{0}{1})$ to a pair of rational numbers with Farey determinant 1, and then $M \in \mathrm{PGL}_2(\mathbb{Z})$. Thus the group of symmetries of the Farey tessellation is $\mathrm{PGL}_2(\mathbb{Z})$.

To describe more precisely these symmetries, we extend the notion of Farey determinant, defined in Chapter 1, Definition 1.1.10, to pairs of elements of $\mathbb{Q}[i]$. Since $\mathbb{Q}[i]$ is an Euclidean domain, any element can be written as a fraction of Gaussian integers $\frac{a}{b}$, uniquely if we require that $\gcd(a,b) = 1$ and that $\mathrm{Real}(b)$ and $\mathrm{Im}(b)$ are nonnegative integers. This is called the reduced form of the element $\frac{a}{b} \in \mathbb{Q}[i]$.

Definition 3.1.15. The *Farey determinant* of two elements $\frac{a}{b}$ and $\frac{c}{d}$, in reduced form in $\mathbb{Q}[i]$, is

$$d_F\left(\frac{a}{b}, \frac{c}{d}\right) = |ad - bc|\,.$$

One can include $\infty = \frac{1}{0}$ in this definition.

Lemma 3.1.16. *For any matrix $M \in \mathrm{GL}_2(\mathbb{Z})$, and any element $z \in \mathbb{Q}[i]$, we have*

$$d_F\left(M \cdot z, M \cdot \overline{z}\right) = d_F(z, \overline{z}).$$

Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$, and write $z = \frac{u}{v} \in \mathbb{Q}[i]$ in reduced form. Since $ad - bc = \pm 1$, we have

$$(au + bv)d - b(cu + dv) = \pm u \text{ and } (au + bv)c - a(cu + dv) = \mp u.$$

But $\gcd(u, v) = 1$, so the fraction $\frac{au+bv}{cu+dv}$ is reduced. Similarly, $\frac{a\overline{u}+b\overline{v}}{c\overline{u}+d\overline{v}}$ is reduced. Thus

$$\begin{aligned} d_F\left(M \cdot \frac{u}{v}, M \cdot \frac{\overline{u}}{\overline{v}}\right) &= |(au + bv)(c\overline{u} + d\overline{v}) - (cu + dv)(a\overline{u} + b\overline{v})| \\ &= |u\overline{v} - v\overline{u}| \\ &= d_F\left(\frac{u}{v}, \frac{\overline{u}}{\overline{v}}\right). \end{aligned}$$

□

Proposition 3.1.17 ([JPRT26])**.** *The set of orientation-reversing involutive symmetries of the Farey tessellation is the set of inversions in geodesics with endpoints in $\mathbb{P}^1(\mathbb{Q})$ of Farey determinant 1 or 2.*
The set of orientation-preserving involutive symmetries of the Farey tessellation is the set of hyperbolic rotations of π around a point $z \in \mathbb{Q}[i] \cap \mathbb{H}$ such that $d_F(z, \overline{z}) = 2$.

Proof. Let I_γ be an inversion in the geodesic γ whose endpoints have Farey determinant 1 or 2.
Consider the involutions $N : z \mapsto -\overline{z}$ and $TN : z \mapsto -\overline{z} + 1$. These are symmetries of the Farey tessellation, and are inversions in the half line $\mathrm{Real}(z) = 0$ and $\mathrm{Real}(z) = \frac{1}{2}$ respectively. Because the action of $\mathrm{PGL}_2(\mathbb{Z})$ is transitive on geodesics whith endoints of Farey determinant 1 (resp. 2), there exists a transformation $M \in \mathrm{PGL}_2(\mathbb{Z})$ such that $I_\gamma = MNM^{-1}$ or $I_\gamma = MTNM^{-1}$. Thus I_γ is a symmetry of the Farey tessellation.
Conversely, let $M \in \mathrm{PGL}_2(\mathbb{Z})$ be an orientation-reversing involutive symmetry of the Farey tessellation. By the general theory of isometries of $\mathbb{H}$, the involution M is an inversion in some geodesic γ. If γ belongs to the Farey tessellation, it is done. If not, then γ must intersect the interior of a Farey triangle, and by conjugating by an element of $\mathrm{PGL}_2(\mathbb{Z})$, we can assume that this triangle is the base triangle Δ_0 delimited by 0, 1 and ∞. Because

M is a symmetry of the Farey tessellation, for each edge that γ crosses, it must cross it orthogonally. The only possible case is if γ is the half line $\text{Real}(z) = \frac{1}{2}$, which is a geodesic whose endpoints have Farey determinant 2.
The case of orientation-preserving involutions can be proven with similar arguments, see [JPRT26, Proposition 3.8]. □

Corollary 3.1.18 ([JPRT26]). *If two distinct rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are exchanged under an orientation-reversing involution, then the fixed points (in $\mathbb{R}$) of this involution are $\frac{a+c}{b+d}$ and $\frac{a-c}{b-d}$.*

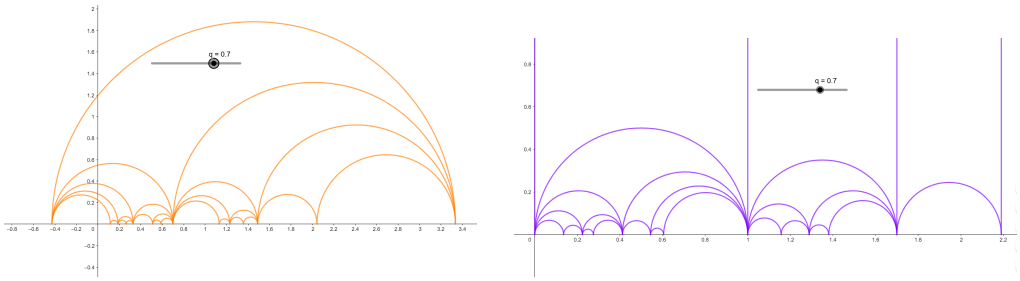
If two distinct rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are exchanged under an orientation-preserving involution, then the fixed points (in $\mathbb{C}$) of this involution are $\frac{a+ci}{b+di}$ and $\frac{a-ci}{b-di}$.

3.2 q-deformed Farey operations

The q-rational numbers can be computed using a q-deformation of the Farey graph, as explained in Chapter 1, Section 3.2. In what follows, we will consider q as a real parameter and not an indeterminate anymore. This shift in the role of the variable q leads to another understanding of the q-Farey graphs, more geometric. The specialization of $q \in \mathbb{R}$ was actually first explored by Bapat, Becker and Licata when they defined left q-rational numbers [BBL23].

3.2.1 The q-Farey tessellations

Let us consider a fixed value for the parameter $q \in (0, 1)$. Then the deformation maps $[\cdot]_q^\sharp$ and $[\cdot]_q^\flat$ associate to any rational number $x \in \mathbb{Q}$ two real numbers $[x]_q^\sharp, [x]_q^\flat$. As before, we often will remove the subscript $_q$ of the notation. In this context, the right (resp. left) q-Farey graph is a graph embedded in $\overline{\mathbb{H}}$ whose vertices are the right (resp. left) q-rational numbers and edges are geodesics.



(a) Left version, $q = 0.7$ (b) Right version, $q = 0.7$

Figure 3.2: Partial representations of q-deformed Farey graphs for $q = 0.7$.

We reproduce here the main properties of q-deformations of the Farey graph, introduced in Section 1.4.2.

Proposition 3.2.1 ([MGO20, Ren24]). *Let $0 \leq \frac{a}{b} < \frac{c}{d}$ be two rational numbers with Farey determinant 1. Denote by $\frac{A^\sharp}{B^\sharp}$ and $\frac{C^\sharp}{D^\sharp}$ the right q-deformations of $\frac{a}{b}$ and $\frac{c}{d}$, and similarly for the left q-deformations. Then*

$$\left[\frac{a+c}{b+d}\right]_q^\sharp = \frac{A^\sharp + q^\alpha C^\sharp}{B^\sharp + q^\alpha D^\sharp},$$

with $\alpha = c_k - 1$*, if the negative continued fraction of* $\frac{a+c}{b+d}$ *is* $[\![c_1, \cdots, c_k]\!]$*. Likewise,*

$$\left[\frac{a+c}{b+d}\right]_q^\flat = \frac{q^\beta A^\flat + C^\flat}{q^\beta B^\flat + D^\flat},$$

where β *is computed from the even positive continued fraction of* $\frac{a+c}{b+d} = [a_1, \cdots, a_{2m}]$*. If* $\frac{a}{b} \neq \frac{0}{1}$*, then* $\beta = a_{2m}$ *and otherwise* $\beta = a_{2m} - 1$*.*

Remark 3.2.2. The power of q in these formulas can be computed using the integers $\varepsilon(x)$ as defined in Chapter 1, Definition 1.4.10. Keeping the notations of the proposition above, we have $\alpha = \varepsilon(\frac{a+c}{b+d}) - \varepsilon(\frac{c}{d})$ and $\beta = \varepsilon(\frac{a+c}{b+d}) - \varepsilon(\frac{a}{b})$.

The convergence properties of q-rational numbers, see Proposition 1.5.7, can be read in the q-deformed Farey graphs. Indeed, given a rational number x, consider $[x]^\sharp$ in the right q-Farey graph. The sequence of the left neighbours of $[x]^\sharp$ then converges, and the limit is exactly $[x]^\flat$. This is explained in details in Section 2.2 of [BBL23]. What Bapat, Becker and Licata actually highlight is that in the right q-Farey graph, the sequence of triangles to the left of a given rational number x converges toward a geodesic, linking $[x]^\flat$ and $[x]^\sharp$.

Besides, for any sequence of rational numbers $(x_n)_n$ converging to an irrational number x, the sequences of left and right q-deformations $([x_n]^\sharp)_n$ and $([x_n]^\flat)_n$ converge toward the same real number, that is $[x]_q$. Hence, irrational numbers are q-deformed as a single value $[x]_q$, whereas rational numbers are q-deformed as geodesics between two real values, $[x]^\flat$ and $[x]^\sharp$. We will come back to this idea in Section 3.3.

3.2.2 Generalized Farey operations

The q-Farey sum was only defined for pairs of rational numbers with Farey determinant 1. They correspond to points in the Farey graph which are at graph distance 1 (they have an edge between them). We will generalize the above proposition to pairs of rationals of Farey graph distance 2.

Remark 3.2.3. Note that the Farey graph distance is not linked in general to the Farey determinant. They coincide only in the case when both are 1. For example, for all $n \in \mathbb{Z}_{\geq 1}$, the pair $(\frac{n}{n+1}, \frac{2}{1})$ has a Farey graph distance of 2, but its Farey determinant is $n + 2$.

There is a nice characterization of pairs with Farey distance at most 2.

Proposition 3.2.4 ([JPRT26])**.** *Consider a pair of rationals* $(\frac{a}{b}, \frac{c}{d})$ *and denote by* d_F *their Farey determinant. The pair is of Farey graph distance at most 2 inside the Farey graph if and only if* $\gcd(a + c, b + d) = d_F$ *or* $\gcd(a - c, b - d) = d_F$*.*

Proof. Assume that $\gcd(a + c, b + d) = d_F$. The Farey sum of $\frac{a}{b}$ and $\frac{c}{d}$ has reduced form $\frac{(a+c)/d_F}{(b+d)/d_F}$, and we can compute the Farey determinant of the pair $(\frac{a}{b}, \frac{a+c}{b+d})$:

$$d_F(\tfrac{a}{b}, \tfrac{a+c}{b+d}) = \frac{a(b+d) - b(a+c)}{d_F} = 1.$$

The same computation shows that $d_F(\frac{c}{d}, \frac{a+c}{b+d}) = 1$. Hence the graph distance of $\frac{a}{b}$ and $\frac{c}{d}$ is at most 2. A similar argument works for the assumption that $\gcd(a - c, b - d) = d_F$.

Conversely, assume that the graph distance is at most 2. If the distance is 1, then the Farey determinant is 1, and $(a+c)b-(b+d)a=1$ shows that $\gcd(a+c,b+d)=1$. A similar argument shows that $\gcd(a-c,b-d)=1$. If the distance is 2, we know that there is a rational number $\frac{x}{y}$ such that $d_F(\frac{x}{y},\frac{a}{b})=1=d_F(\frac{x}{y},\frac{c}{d})$. This leads to a system in (x,y) with solution $x=\frac{\pm a\pm c}{d_F}$ and $y=\frac{\pm b\pm d}{d_F}$. Since $\frac{x}{y}$ is a reduced fraction, we get that either $\gcd(a+c,b+d)=d_F$ or $\gcd(a-c,b-d)=d_F$. □

Motivated by this characterization, we define a Farey difference.

Definition 3.2.5. Let $(\frac{a}{b},\frac{c}{d})$ be a pair of rational numbers in reduced form. Their *Farey difference* is the extended rational number

$$\frac{a}{b}\ominus_F\frac{c}{d}=\frac{a-c}{b-d}\in\mathbb{P}^1(\mathbb{Q}).$$

Remark 3.2.6. The fraction obtained by the Farey difference may not be in reduced form, except when the pair $(\frac{a}{b},\frac{c}{d})$ has Farey determinant 1, exactly as for the Farey sum.

We are going to give a q-deformed formula of the Farey difference as well as for the Farey sum. For this, we need the q-deformed Farey determinant introduced in Chapter 1, Section 1.4.3. Given a rational number $\frac{u}{v}$ in reduced form, denote by $\frac{U^\sharp}{V^\sharp}$ and $\frac{U^\flat}{V^\flat}$ its right and left q-deformations, with $U^\sharp$ and $V^\sharp$ (resp. $U^\flat$ and $V^\flat$) two coprime polynomials in q, and $V^\sharp$, $V^\flat$ having positive coefficients. The q-Farey determinants are as follows.

$$d_F^{\sharp\sharp}\left(\tfrac{a}{b},\tfrac{c}{d}\right)=B^\sharp C^\sharp-A^\sharp D^\sharp\quad,\quad d_F^{\sharp\flat}\left(\tfrac{a}{b},\tfrac{c}{d}\right)=B^\sharp C^\flat-A^\sharp D^\flat$$

$$d_F^{\flat\sharp}\left(\tfrac{a}{b},\tfrac{c}{d}\right)=B^\flat C^\sharp-A^\flat D^\sharp\quad,\quad d_F^{\flat\flat}\left(\tfrac{a}{b},\tfrac{c}{d}\right)=\frac{B^\flat C^\flat-A^\flat D^\flat}{q^2-q+1}.$$

It was proven in Proposition 1.4.32 and Theorem 1.4.33 that these q-Farey determinants are polynomials with positive coefficients.

Theorem 3.2.7 ([JPRT26])**.** *Let $(\frac{a}{b},\frac{c}{d})$ be a pair of rationals, and denote by d_F their Farey determinant. Assume that $\gcd(a+c,b+d)=d_F$. Then there are integers α,β such that*

$$\left[\frac{a}{b}\oplus_F\frac{c}{d}\right]_q^\sharp=\frac{(q^\alpha A^\sharp+q^\beta C^\sharp)/d_F^{\sharp\sharp}}{(q^\alpha B^\sharp+q^\beta D^\sharp)/d_F^{\sharp\sharp}},$$

where the fraction on the right hand side is reduced.
If we assume instead that $\gcd(a-c,b-d)=d_F$, then there exist integers α and β such that

$$\left[\frac{a}{b}\ominus_F\frac{c}{d}\right]_q^\sharp=\frac{(q^\alpha A^\sharp-q^\beta C^\sharp)/d_F^{\sharp\sharp}}{(q^\alpha B^\sharp-q^\beta D^\sharp)/d_F^{\sharp\sharp}},$$

where the fraction on the right hand side is again reduced.

The idea of the proof is to show that the property is invariant under the $\mathrm{PSL}_2(\mathbb{Z})$-action, which allows to reduce to the pair $(\frac{1}{0},\frac{-1}{n})$. Finding an explicit formula for α and β seems challenging.

Proof. We focus on the Farey sum, the proof for the Farey difference is the same. Put $\frac{r}{s} = \frac{a}{b} \oplus_F \frac{c}{d}$, i.e. $r = (a+c)/d_F$ and $s = (b+d)/d_F$. We have to show that there are integers α and β such that

$$\begin{aligned} d_F^{\sharp\sharp} R^\sharp &= q^\alpha A^\sharp + q^\beta C^\sharp; \\ d_F^{\sharp\sharp} S^\sharp &= q^\alpha B^\sharp + q^\beta D^\sharp. \end{aligned} \tag{3.2}$$

Let us show that these equations are invariant under the deformed $\mathrm{PSL}_2(\mathbb{Z})$-action. By Proposition 1.4.31, we know that the q-Farey determinant is invariant, up to some power of q which can be absorbed in the integers α and β.
Then the crucial fact is that the classical Farey sum is equivariant with respect to any Möbius transformation $M \in \mathrm{PSL}_2(\mathbb{Z})$. Hence the action of S or T has the same effect on $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{r}{s}$. Let us compute the effect of the transformations S_q and T_q on a q-rational number $\frac{X^\sharp}{Y^\sharp}$. We write $(X^\sharp, Y^\sharp)$ to identify without ambiguity the numerator and the denominator.

$$(X^\sharp, Y^\sharp) \overset{T_q}{\longmapsto} (qX^\sharp + Y^\sharp, Y^\sharp) \text{ or } (X^\sharp + q^{-1}Y^\sharp, q^{-1}Y^\sharp),$$

$$(X^\sharp, Y^\sharp) \overset{S_q}{\longmapsto} (-Y^\sharp, qX^\sharp) \text{ or } (-q^{-1}Y^\sharp, X^\sharp).$$

Applying this to $(A^\sharp, B^\sharp)$, $(C^\sharp, D^\sharp)$ and $(R^\sharp, S^\sharp)$, we see that Equations (3.2) are still satisfied after the actions of S_q and T_q, changing α or β if necessary.

Using the $\mathrm{PSL}_2(\mathbb{Z})$-invariance, we can reduce to the case where $\frac{a}{b} = \frac{1}{0}$. Then $\frac{c}{d} = \frac{k}{n}$ with $n = d_F$. By the assumption on the greatest common divisor, we see that $k \equiv -1 \mod n$. Using T, we can assume $k = -1$. Then $\alpha = \beta = 0$ satisfy Equations (3.2). Indeed, $[\infty]^\sharp = \frac{1}{0}$ and $[-\frac{1}{n}]^\sharp = \frac{-1}{q[n]^\sharp}$. Hence $d_F^{\sharp\sharp} = q[n]^\sharp$. Finally, $\frac{1}{0} \oplus_F \frac{-1}{n} = \frac{0}{1}$, with deformation $[0]^\sharp = \frac{0}{1}$. □

Remark 3.2.8. The same theorem holds in the left version. If $\gcd(a+c, b+d) = d_F$, there are integers α, β such that

$$\left[\frac{a}{b} \oplus_F \frac{c}{d}\right]^\flat = \frac{(q^\alpha A^\flat + q^\beta C^\flat)/d_F^{\flat\flat}}{(q^\alpha B^\flat + q^\beta D^\flat)/d_F^{\flat\flat}},$$

and same for the Farey difference under the assumption $\gcd(a-c, b-d) = d_F$.
More generally one can even show, using the same sketch of arguments, that under the assumption $\gcd(a+c, b+d) = d_F$,

$$\left[\frac{a}{b} \oplus_F \frac{c}{d}\right]^\sharp = \frac{(q^\alpha A^\sharp + q^\beta C^\flat)/d_F^{\sharp\flat}}{(q^\alpha B^\sharp + q^\beta D^\flat)/d_F^{\sharp\flat}},$$

Note that the integers α and β depend on the different cases. Similarly,

$$\left[\frac{a}{b} \oplus_F \frac{c}{d}\right]^\flat = \frac{(q^\alpha A^\sharp + q^\beta C^\sharp)/d_F^{\sharp\sharp}}{(q^\alpha B^\sharp + q^\beta D^\sharp)/d_F^{\sharp\sharp}},$$

And under the assumption $\gcd(a-c, b-d) = d_F$, same formulas hold with the Farey difference instead.

Example 3.2.9. To illustrate the difficulty of finding a general formula to compute the powers of q appearing in the q-Farey operations, we give two examples of pairs $(\frac{a}{b}, \frac{c}{d})$ with same Farey determinant and same integers $\varepsilon(\frac{a}{b}), \varepsilon(\frac{c}{d})$ (see (3.3)).
The first pair is $(\frac{a}{b}, \frac{c}{d}) = (3, 4)$, of Farey determinant 1. We have $\varepsilon(3) = 2$ and $\varepsilon(4) = 3$. And

$$\left[\frac{7}{2}\right]^\sharp = \frac{[3]^\sharp + q[4]^\sharp}{1+q},$$

so $\alpha = 0$ and $\beta = 1$.
The second pair we consider is $(\frac{3}{2}, \frac{4}{3})$ which has Farey determinant 1 and $\varepsilon(\frac{3}{2}) = 2$, $\varepsilon(\frac{4}{3}) = 3$. We have

$$\left[\frac{7}{5}\right]^\sharp = \frac{q^2[3]^\sharp + [4]^\sharp}{q^2[2]^\sharp + [3]^\sharp},$$

so $\alpha = 2$ and $\beta = 0$.

3.3 Quantum rationals as disks

We know that any rational number gives rise to a limit geodesic in the right q-Farey graph, with left endpoint $[x]^\flat$ and right endpoint $[x]^\sharp$. It was suggested by Bapat, Becker and Licata [BBL23, Section 2.5] that the "true q-analogue" of a rational number is the entire geodesic and not just its endpoints. We adopt this point of view and explore some of its consequences. To get a more symmetric picture, we use the complex conjugation to work in the entire plane $\mathbb{C}$ viewed as the union of the closed hyperbolic plane $\overline{\mathbb{H}}$ and its horizontal reflection.

3.3.1 Geodesics and disks

Definition 3.3.1 ([JPRT26])**.** Let $q \in (0, 1)$. To each $x \in \mathbb{P}^1(\mathbb{Q})$, we associate the unique closed disk in $\mathbb{P}^1(CC)$ orthogonal to the real axis and crossing it at $[x]^\flat$ and $[x]^\sharp$. We denote this disk by $[x]$.
The union of all the disks $[x]$, for $x \in \mathbb{P}^1(\mathbb{Q})$ is denoted by $\mathcal{Q} \subset \mathbb{C}$.

Note that for $x = \infty$, the disc $[\infty]$ has infinite radius. It is then a half-plane, delimited by the vertical line at $\frac{1}{1-q}$.

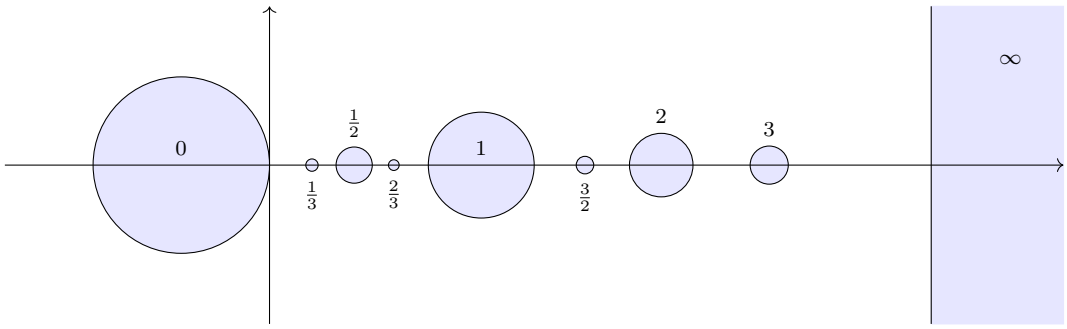


Figure 3.3: The disks associated to rational numbers of depth at most 3 in the Farey graph. The parameter q is fixed at 0.6.

The well-orderedness of q-rational numbers, see Corollary 1.4.34, implies that the disks we get are disjoint. We refer to [JPRT26] for another proof of this fact, using geometric methods.

Proposition 3.3.2. *For any $x, y \in \mathbb{P}^1(\mathbb{Q})$ with $x \neq y$, the disks $[x]$ and $[y]$ are disjoint.*

Remark 3.3.3. The well-orderedness of q-rationals might come as a surprise on the first sight, since rational numbers are already dense in $\mathbb{R}$. For $x \in \mathbb{Q}$, the interval $[[x]^\flat, [x]^\sharp]$ does not contain x in general. Hence the construction of q-rationals, step by step via Farey addition, is similar to the construction of a Cantor set.

For two rational numbers x, y, we denote by $\gamma([x],[y])$ the unique geodesic orthogonal to both $[x]$ and $[y]$. From Proposition 3.1.17 and the $\mathrm{PGL}_2(\mathbb{Z})$-symmetry of $\mathcal{Q}$, we get:

Corollary 3.3.4. *For $x, y \in \mathbb{P}^1(\mathbb{Q})$, if $d_F(x,y) \in \{1,2\}$, the inversion with respect to $\gamma([x],[y])$ is a symmetry of $\mathcal{Q}$.*

More details about the surface $\mathbb{H}\backslash\mathcal{Q}$ and its quotient by the deformed action of $\mathrm{PSL}_2(\mathbb{Z})$ can be found in [JPRT26]. This deformed modular surface has also independently been described by Simon [Sim25, Section 3.2].

We now compute diameters of q-rationals viewed as disks. Recall from Chapter 1 that we associate a positive integer $\varepsilon(x)$ to each rational number x (see Definition 1.4.10).

$$\varepsilon(x) = \begin{cases} |a_1| - 1 \text{ if } x \in \mathbb{Z}_{\leq 0}, \\ |a_1| + a_2 + \ldots + a_{2m} - 1 \text{ else,} \end{cases} \tag{3.3}$$

if $x = [a_1, \cdots, a_{2m}]$ is the even positive continued fraction of x.

Lemma 3.3.5. *Let $x = \frac{a}{b} \in \mathbb{Q}$, and let $\frac{A^\flat}{B^\flat}$ and $\frac{A^\sharp}{B^\sharp}$ be its left and right q-deformations. Then $[x]$ is a disk of diameter given by*

$$\ell_q(x) = |1-q| \, \frac{q^{\varepsilon(x)}}{B^\sharp B^\flat}, \tag{3.4}$$

Proof. Suppose $x \notin \mathbb{Z}_{\leq 0}$. We have

$$\ell_q(x) = \left| [x]^\sharp - [x]^\flat \right| = \frac{\left| A^\sharp B^\flat - A^\flat B^\sharp \right|}{B^\sharp B^\flat}.$$

Consider the q-matrix $M_{+,q}(x)$ associated to x (see Definition 1.20). By Proposition 1.4.6 and Corollary 1.4.65, it satisfies

$$M_{+,q}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = q^{\min(0,\lfloor x \rfloor)} \begin{pmatrix} qA^\sharp \\ qB^\sharp \end{pmatrix} \text{ and } M_{+,q}(x) \begin{pmatrix} 1 \\ 1-q \end{pmatrix} = q^{\min(0,\lfloor x \rfloor)} \begin{pmatrix} A^\flat \\ B^\flat \end{pmatrix}.$$

Then

$$\det\left(M_q \begin{pmatrix} 1 & 1 \\ 0 & 1-q \end{pmatrix} \right) = q^{2\min(0,\lfloor x \rfloor)} \det \begin{pmatrix} qA^\sharp & A^\flat \\ qB^\sharp & B^\flat \end{pmatrix} = q^{2\min(0,\lfloor x \rfloor)+1}(A^\sharp B^\flat - A^\flat B^\sharp).$$

But $\det(M_q) = q^{\varepsilon(x)+2\min(0,\lfloor x \rfloor)+1}$, and hence the formula.
The proof is analogue for the case $x \in \mathbb{Z}_{\leq 0}$. □

Example 3.3.6. When $x = n \in \mathbb{N}^*$, we have $\varepsilon(n) = n - 1$ and the diameter of $[n]$ looks like

$$\ell_q(n) = |1 - q|\, q^{n-1}.$$

Hence the sequence of diameters of the q-integers decrease as a geometric progression of ratio q.

As a consequence of this explicit formula for the diameter, we see that for q close to 1, the Taylor expansion of Equation (3.4) gives

$$\ell_q\left(\frac{a}{b}\right) \sim_{q\to 1} \frac{|1-q|}{b^2}.$$

In other words: the radius of the disk associated to $\frac{a}{b}$ is $(1 - q)$ times the radius of the Ford circle.

The notation of $\ell_q(x)$ is is the one used by Etingof, who speaks of jumps for differences between the left and right deformations of a rational [Eti25, Section 6]. Using analytic considerations, he computes the sum of jumps of all rationals for q in some open subset of $\mathbb{C}$.

3.3.2 Extremal values of q

The limit case $q = 0$ is worth taking a closer look. As displayed in Figure 3.4, almost all the disks collapse, and there only remain $[0]$, $[1]$ and $[\infty]$.

Indeed, for any $0 < x < 1$, the numerators of $[x]_q^\flat$ and $[x]_q^\sharp$ have valuation > 1, and the denominators have valuation 0 (see Corollary 1.4.65). Therefore $[x]_q^\sharp, [x]_q^\flat \underset{q\to 0}{\longrightarrow} 0$.

On the other hand, if $1 < x < \infty$, then $[x]^\sharp, [x]^\flat \underset{q\to 0}{\longrightarrow} 1$.

Finally the negative rational numbers $x < 0$ produce disks exploding to ∞.

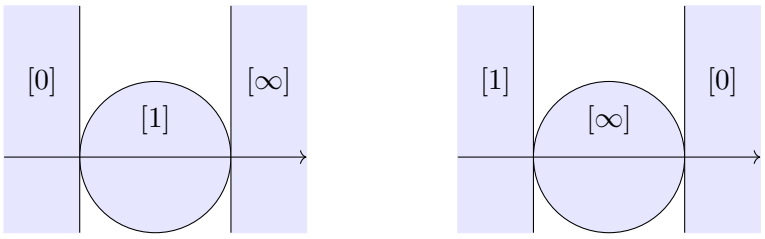


Figure 3.4: The q-deformed rational numbers at extremal values $q = 0$ (on the left) and $q = \infty$ (on the right)

The dual situation occurs when $q \to \infty$. Since the numerator and denominator of the (left or right) deformation of any rational number $0 < x < 1$ have same degree, we have $[x]_q^\sharp, [x]_q^\flat \underset{q\to\infty}{\longrightarrow} 1$. If $1 < x < \infty$, then $[x]_q^\sharp, [x]_q^\flat \underset{q\to\infty}{\longrightarrow} \infty$. And on the contrary if $x < 0$, then the disk $[x]$ converges toward 0. Geometrically, the picture we get is the same as the one for $q = 0$, up to a rotation of labels of the three disks. The duality $q \mapsto \frac{1}{q}$ is indeed a hyperbolic rotation around $\sigma = \frac{1+\sqrt{3}}{2}$.

These observations are the starting point of a work in progress with Olga Paris-Romaskevich and Alexander Thomas aiming at defining q-complex numbers geometrically.

3.4 Classical Springborn operations

In this section, we present a quadratic version of the Farey operations, which we call in [JPRT26] the Springborn operations, because the Springborn sum has been used by Springborn in [Spr24].

3.4.1 Motivation: Homothetic centers

Consider two circles C_1 and C_2. Denote by c_1 and c_2 their centers and by r_1 and r_2 their radii.

Definition 3.4.1. The *inner homothety center* of C_1 and C_2, denoted by $i(C_1, C_2)$, is the fixed point of the unique homothety with negative factor exchanging C_1 and C_2. Similarly, the *outer homothety center*, denoted by $e(C_1, C_2)$, is the fixed point of the unique homothety with positive factor exchanging C_1 and C_2.

In the case when C_1 and C_2 have disjoint interiors, $i(C_1, C_2)$ (resp. $e(C_1, C_2)$) is the intersection of the inner (resp. outer) common tangents of C_1 and C_2, see Figure 3.5.

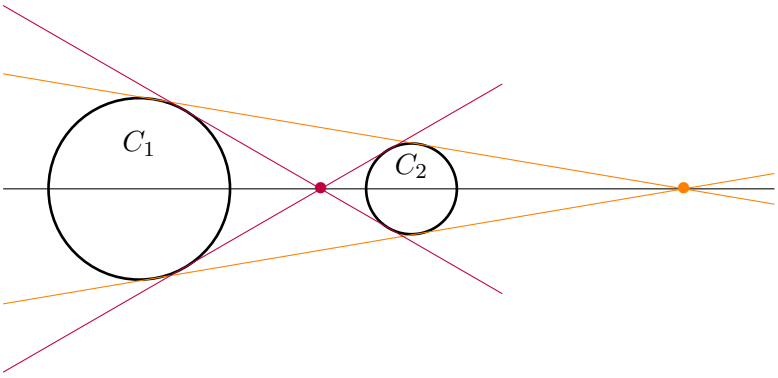


Figure 3.5: Inner (in purple) and outer (in orange) homothety centers.

Observation 3.4.2. For many pairs $\left(\frac{a}{b}, \frac{c}{d}\right) \in \mathbb{Q}^2$, we find

$$i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \left[\frac{ab+cd}{b^2+d^2}\right]^\sharp \text{ and } e\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \left[\frac{ab-cd}{b^2-d^2}\right]^\flat .$$

One of the main result of this work is to prove this observation for some pairs satisfying arithmetic conditions, that we call regular pairs. Theorem 3.5.2 covers the particular case of pairs with Farey determinant 1 or 2, and then Theorem 3.5.10 gives the more general setting.

Remark 3.4.3. The formula $\frac{ab+cd}{b^2+d^2}$ first appears in the study of Ford circles [For38]. It seems that the first time it was used as an iterative operation was recently by Springborn [Spr24] in the context of Diophantine approximation of rationals. Springborn defines the notion of Markov fractions, which can be obtained as iterations of this formula on the pair $(\frac{0}{1}, \frac{1}{1})$. Then, this formula was explored in the subsequent work of Veselov [Ves25, Equation (1)]. We study q-deformed Markov fractions in Section 3.6.

3.4.2 Regular pairs and Springborn operations

Motivated by Observation 3.4.2, we define the Springborn sum and difference.

Definition 3.4.4. Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rationals in reduced form (or equal to $\infty = \frac{1}{0}$). The *Springborn sum* and *Springborn difference* respectively of this pair of rationals is

$$\frac{a}{b} \oplus_S \frac{c}{d} = \frac{ab+cd}{b^2+d^2} \quad \text{and} \quad \frac{a}{b} \ominus_S \frac{c}{d} = \frac{ab-cd}{b^2-d^2}.$$

Note that these expressions are in general not reduced.

Remark 3.4.5. The Springborn operations are *not* equivariant with respect to Möbius transformations. For example, $S \cdot (\frac{1}{1} \oplus_S \frac{3}{2}) = -\frac{5}{7}$ but $(-\frac{1}{1}) \oplus_S (-\frac{2}{3}) = -\frac{7}{10}$, where $S : z \mapsto -\frac{1}{z}$.

We define now a family of pairs for which the reduced forms of Springborn operations is easy to compute.

Definition 3.4.6. A pair $(\frac{a}{b}, \frac{c}{d}) \in \mathbb{Q}^2$ with Farey determinant $d_F = |ad-bc|$ is said to be *inner regular* if

$$\gcd(ab+cd, b^2+d^2, a^2+c^2) = d_F.$$

Similarly, the pair is *outer regular* if $\gcd(ab-cd, b^2-d^2, a^2-c^2) = d_F$.

Remark 3.4.7. There are pairs of rational numbers that are neither inner-regular nor outer-regular, for example $(\frac{1}{3}, \frac{2}{9})$.

The following lemma will be useful to check regularity.

Lemma 3.4.8. *For two rational numbers $\frac{a}{b}, \frac{c}{d}$, the quantities $\gcd(a^2+c^2, ab+cd)$ and $\gcd(b^2+d^2, ab+cd)$ divide d_F.*
If moreover $\gcd(b,d) = 1$, then $(\frac{a}{b}, \frac{c}{d})$ is inner regular if and only if $d_F \mid b^2+d^2$.
A similar statement holds for the outer regular case.

Proof. We have the following identities:

$$\begin{cases} b(ab+cd) = a(b^2+d^2) - d\,d_F \\ a(ab+cd) = b(a^2+c^2) + c\,d_F, \\ (a^2+c^2)(b^2+d^2) = (ab+cd)^2 + d_F^2. \end{cases} \tag{3.5}$$

The third one implies the first statement of the lemma.
Now suppose $\gcd(b,d) = 1$. We only have to prove that $d_F \mid b^2+d^2$ implies inner regularity. By the first relation in (3.5), we see that d_F divides $b(ab+cd)$, but $\gcd(b,d) = 1$ so d_F and b are coprime, thus d_F divides $\gcd(b^2+d^2, ab+cd)$. By the first part of the lemma, we get $\gcd(b^2+d^2, ab+cd) = d_F$. Finally, the second relation in (3.5) implies that d_F divides a^2+c^2, hence the inner regularity condition is satisfied. □

Let us now see some special cases.

Example 3.4.9. When $d_F = 1$, the pair is both inner and outer regular by the last part of Lemma 3.4.8 ($d_F = 1$ implies $\gcd(b,d) = 1$). Therefore a pair of neighbours in the Farey graph is inner-regular and outer-regular.

Example 3.4.10. In the special case $d_F = 2$, the pair is also inner and outer regular. Indeed, if $\gcd(b,d) = 1$, then again by Lemma 3.4.8 it is sufficient to show that $2 \mid b^2 + d^2$. If b is even and d is odd, then a is also odd, which contradicts $d_F = ad - bc = 2$. Hence both b and d are odd, so $2 \mid b^2 + d^2$. Now if $\gcd(b,d) > 1$, then $\gcd(b,d) = 2$ and a, c are both odd. Put $b = 2b'$ and $d = 2d'$. It is then easy to see that $ab' + cd'$ is odd, hence $\gcd(ab + cd, b^2 + d^2)/2 = \gcd(ab' + cd', b'^2 + d'^2) = 1$, where the last equality comes from Lemma 3.4.8. Therefore the pair is inner regular. Similar arguments show the same for the outer case.

Proposition 3.4.11 ([JPRT26])**.** *The set of inner regular pairs is invariant under the modular group action. A set of representatives is given by* $(\frac{1}{0}, \frac{k}{n})$ *where* $n \mid k^2+1$, $0 \leq k < n$, *and* $\gcd(k,n) = 1$.

Similarly, the set of outer regular pairs is invariant under $\mathrm{PSL}_2(\mathbb{Z})$ *and a set of representatives is given by* $(\frac{1}{0}, \frac{k}{n})$ *where* $n \mid k^2 - 1$, *and* $0 \leq k < n$.

Proof. Applying S to a pair $(\frac{a}{b}, \frac{c}{d})$ exchanges a with b and c with d. Hence it does not change the regularity condition.
Applying T changes a into $a+b$ and c into $c+d$, but keeps b and d the same. By elementary operations in the greatest common divisor, we see that regularity is preserved under T.
By Proposition 1.4.30, the Farey determinant d_F is invariant under $\mathrm{PSL}_2(\mathbb{Z})$-action with representative $(\frac{1}{0}, \frac{k}{n})$, with $0 \leq k < n$, for a pair of Farey determinant n. Inner regularity implies that $\gcd(kn, n^2, 1 + k^2) = n$. This is equivalent to $n \mid k^2 + 1$.
Similarly, outer regulartiy implies that $\gcd(kn, n^2, -1 + k^2) = n$, which is equivalent to $n \mid k^2 - 1$. □

Remark 3.4.12. According to this Proposition 3.4.11, inner regular pairs can not have any value for the Farey determinant. The integer n appears as Farey determinant of an inner regular pair if and only if -1 is a square in $\mathbb{Z}/n\mathbb{Z}$. For prime n, this is equivalent to requiring that $n \equiv 1 \mod (4)$. We give below the list of the first representatives of inner regular pairs for small values of n.

$$\left(\frac{1}{0}, \frac{0}{1}\right); \left(\frac{1}{0}, \frac{1}{2}\right); \left(\frac{1}{0}, \frac{2}{5}\right), \left(\frac{1}{0}, \frac{3}{5}\right); \left(\frac{1}{0}, \frac{3}{10}\right); \left(\frac{1}{0}, \frac{7}{10}\right); \left(\frac{1}{0}, \frac{9}{10}\right); \left(\frac{1}{0}, \frac{5}{13}\right); \left(\frac{1}{0}, \frac{8}{13}\right) \ldots$$

3.4.3 Geometric interpretations

We give a list of geometric constructions giving the Springborn addition and difference.

Lemma 3.4.13. *We have*

$$\frac{a}{b} \oplus_S \frac{c}{d} = \frac{1}{2}\left(\frac{a+ic}{b+id} + \frac{a-ic}{b-id}\right) = \mathrm{Re}\left(\frac{a+ic}{b+id}\right)$$
$$\frac{a}{b} \ominus_S \frac{c}{d} = \frac{1}{2}\left(\frac{a+c}{b+d} + \frac{a-c}{b-d}\right).$$

The four points $\left(\frac{a}{b}, \frac{c}{d}, \frac{a}{b} \oplus_F \frac{c}{d}, \frac{a}{b} \ominus_F \frac{c}{d}\right)$ are harmonic, i.e. their cross-ratio is -1. A simple computation gives the analogue for Springborn operations.

Proposition 3.4.14 ([JPRT26])**.** *Let* $\frac{a}{b} < \frac{c}{d}$ *be two rational numbers. The Springborn sum* $\frac{a}{b} \oplus_S \frac{c}{d}$ *divides the segment* $[\frac{a}{b}, \frac{c}{d}]$ *into parts of ratio* $b^2 : d^2$.
Likewise, the ratio of the distance of $\frac{a}{b}$ *and* $\frac{c}{d}$ *to the Springborn difference* $\frac{a}{b} \ominus_S \frac{c}{d}$ *is* $d^2 : b^2$.
In particular, the four points $\left(\frac{a}{b}, \frac{c}{d}, \frac{a}{b} \oplus_S \frac{c}{d}, \frac{a}{b} \ominus_S \frac{c}{d}\right)$ *are harmonic.*

Other geometric interpretations relies on the following explicit formula for the inner and outer homothety points of two circles.

Proposition 3.4.15 ([JPRT26]). *For two circles C_1 and C_2, with centers c_1 and c_2 and radii r_1 and r_2 respectively, we have:*

$$i(C_1, C_2) = \frac{r_1 c_2 + r_2 c_1}{r_1 + r_2} \quad \text{and} \quad e(C_1, C_2) = \frac{r_1 c_2 - r_2 c_1}{r_1 - r_2}.$$

We can recover the Springborn operations using Ford circles.

Definition 3.4.16. The *Ford circle* $F_{a/b}$ associated to a rational number $\frac{a}{b}$ is the circle with center $\left(\frac{a}{b}, \frac{1}{2b^2}\right)$ and radius $\frac{1}{2b^2}$.

Proposition 3.4.17 ([JPRT26]). *For two rational numbers $\frac{a}{b}, \frac{c}{d}$, we have*

$$\frac{a}{b} \ominus_S \frac{c}{d} = e(F_{a/b}, F_{c/d}) \quad \text{and} \quad \frac{a}{b} \oplus_S \frac{c}{d} = \mathrm{Re}(i(F_{a/b}, F_{c/d})).$$

In particular, if $d_F(\frac{a}{b}, \frac{c}{d}) = 1$, the two Ford circles are tangent and the x-coordinate of the contact point is given by $\frac{ac+bd}{b^2+d^2}$, as already stated in the original paper by Ford [For38].

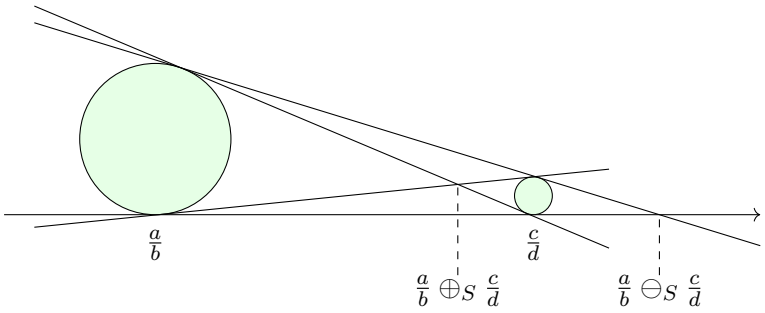


Figure 3.6: Ford circles associated to $\frac{a}{b}$ and $\frac{c}{d}$, and inner and outer homothety centers

Proof. We simply use the explicit formulas for inner and outer homothety center. We get

$$i(F_{a/b}, F_{c/d}) = \frac{\frac{1}{2b^2}(\frac{c}{d} + \frac{i}{2d^2}) + \frac{1}{2d^2}(\frac{a}{b} + \frac{i}{2b^2})}{\frac{1}{2b^2} + \frac{1}{2d^2}} = \frac{ab + cd}{b^2 + d^2} + \frac{i}{b^2 + d^2}.$$

Similarly, we get $e(F_{a/b}, F_{c/d}) = \frac{ab-cd}{b^2-d^2}$. □

It turns out that in the proof of Proposition 3.4.17, we have only used the x-coordinate of the center of the Ford circles (and the radius). So we can move them vertically without changing the result.

Proposition 3.4.18 ([JPRT26]). *Denote by $C_{a/b}$ the circle with center $(\frac{a}{b}, 0)$ and radius $\frac{1}{2b^2}$. Then*

$$\frac{a}{b} \oplus_S \frac{c}{d} = i(C_{a/b}, C_{c/d}) \quad \text{and} \quad \frac{a}{b} \ominus_S \frac{c}{d} = e(C_{a/b}, C_{c/d}).$$

We are grateful to Boris Springborn, who mentioned an equivalent formulation using hyperbolic involutions.

Proposition 3.4.19 ([JPRT26]). *Let C_1, C_2 be two circles centered on the real line, with disjoint interiors.*
(i) The outer homothety center $e(C_1, C_2)$ is the image of ∞ under the unique orientation-reversing isometry of $\mathbb{H}$ exchanging C_1 and C_2.
(ii) Similarly, the inner homothety center $i(C_1, C_2)$ is the image of ∞ under the unique orientation-preserving isometry of $\mathbb{H}$ exchanging C_1 and C_2.

Proof. The space of orientation-reversing isometries of $\mathbb{H}$ consists of inversions in geodesics (circle inversions or axial reflections from Euclidean viewpoint).
Suppose that there is an inversion I_e with respect to a circle C_e exchanging C_1 and C_2. Denote by M_e the midpoint of C_e. Since the image P' of any point P under the inversion I_e is on PM_e, we see that the two tangent lines from M_e to C_2 are also tangent to C_1. Hence $M_e = e(C_1, C_2)$. The radius is then uniquely determined. Conversely, this data determines C_e uniquely. Since an inversion exchanges the center of the circle with infinity, we get $e(C_1, C_2) = M_e = I_e(\infty)$.

For the inner homothety center, consider the geodesic γ_3 between C_1 and C_2 (which exists since C_1 and C_2 have disjoint interiors). This geodesic is part of a circle C_3 which is orthogonal to C_1 and C_2. Denote by I_3 the inversion in C_3. This inversion fixes C_1 and C_2, so also C_e. Hence C_3 is orthogonal to C_e. Consider the composition $I_i = I_e \circ I_3$. Since C_e is orthogonal to C_3, we also have $I_i = I_3 \circ I_e$. Clearly I_i is an orientation-preserving isometry of $\mathbb{H}$. It also exchanges C_1 with C_2. Finally,

$$I_i(\infty) = I_3(I_e(\infty)) = I_3(e(C_1, C_2)) = i(C_1, C_2),$$

where we used that $i(C_1, C_2), e(C_1, C_2)$ together with $C_3 \cap \mathbb{R}$ form four harmonic points, which can be proven by explicit computations. □

3.4.4 Reversion and iteration of Springborn operations

We can ask whether any rational number $\frac{r}{s}$ can be represented as the Springborn sum of two others. We give partial answer, focusing on regular pairs. The results of this paragraph are proven in [JPRT26].

Proposition 3.4.20 ([JPRT26]). *Consider the equation $\frac{a}{b} \oplus_S \frac{x}{y} = \frac{r}{s}$, where a, b, r, s are given and x, y are unknowns. Assuming the pair $(\frac{a}{b}, \frac{x}{y})$ is inner-regular, this equation has solution*

$$x = -ar + \frac{b(1+r^2)}{s} \quad \text{and} \quad y = br - as\ .$$

Hence the equation admits a regular solution if and only if $s \mid b(1+r^2)$ and $\gcd(x, y) = 1$. Similarly, $\frac{a}{b} \ominus_S \frac{x}{y} = \frac{r}{s}$ has an outer-regular solution $x = ar + \frac{b(1-r^2)}{s}$ and $y = as - br$, which is well-defined if and only if $s \mid b(r^2 - 1)$ and $\gcd(x, y) = 1$ and $b \neq y$.

Example 3.4.21. Consider $r = 5, s = 2$ and a, b arbitrary (coprime). Then $y = 5b - 2a$ and $x = -5a + 13b$ and a direct computation gives that $\gcd(x, y) = \gcd(a, b) = 1$. Hence $\frac{5}{2}$ can be written as Springborn sum in infinitely many ways.

The next proposition shows that all rational numbers are attained by the Springborn sum and also by the Springborn difference.

Proposition 3.4.22 ([JPRT26]). *Let $\frac{r}{s}$ be a rational number in reduced form. Let α and β be integers satisfying the Bézout identity $\alpha r + \beta s = 1$. Then*

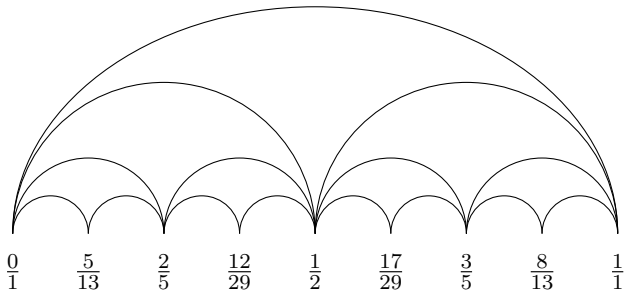
$$\frac{\alpha}{s} \oplus_S \frac{r^2 + \beta s}{s(r - \alpha)} = \frac{r}{s}.$$

Similarly, $\frac{-\alpha}{s} \ominus_S \frac{r^2 - \beta s}{s(r+\alpha)} = \frac{r}{s}$.

The following proposition gives the conditions under which we can iterate the Springborn sum or difference.

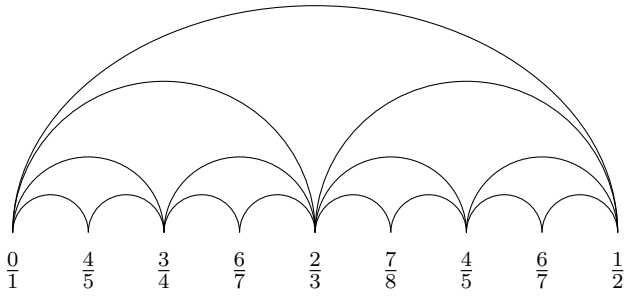
Proposition 3.4.23 ([JPRT26]). *Let $\left(\frac{a}{b}, \frac{c}{d}\right)$ be an inner regular pair with $\gcd(b, d) = 1$. If in addition $b \mid a^2 + 1$ and $d \mid c^2 + 1$, then the pairs $\left(\frac{a}{b}, \frac{a}{b} \oplus_S \frac{c}{d}\right)$ and $\left(\frac{a}{b} \oplus_S \frac{c}{d}, \frac{c}{d}\right)$ are inner regular.*
Similarly, if $\left(\frac{a}{b}, \frac{c}{d}\right)$ is outer regular, $\gcd(b, d) = 1$, $b \mid a^2 - 1$ and $d \mid c^2 - 1$, then one can iterate the Springborn difference.

Example 3.4.24. Starting from $(\frac{0}{1}, \frac{1}{1})$ with Springborn addition, we get the Markov fractions described in the initial paper by Springborn [Spr24]. Up to 3 iterations this gives the following 9 numbers that we represent via a graph:



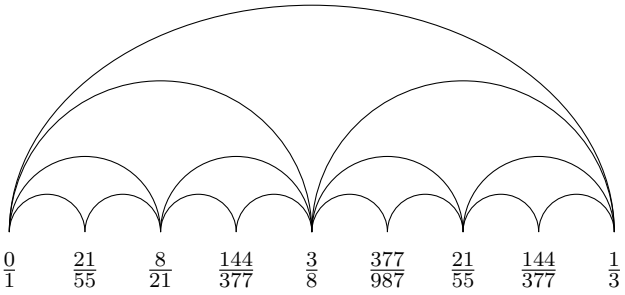
We draw the edges of the graph as curves in order to better represent its combinatorial structure. The same is done in the following two examples.

Example 3.4.25. Starting from $(\frac{0}{1}, \frac{1}{2})$ with Springborn difference, we get fractions of the form $\frac{n}{n+1}$. Up to 3 iterations this gives:



Note that the order is not respected, and that the sequence is not injective.

Example 3.4.26. Starting from $(\frac{0}{1}, \frac{1}{3})$ with Springborn difference, we get all companions of $\frac{0}{1}$ as described in Springborn's original paper, which are quotients of Fibonacci numbers. Up to 3 iterations this gives:



3.5 q-deformed Springborn operations

We use our geometric picture of q-rationals, seen as circles, and apply the Springborn operations to them. For regular pairs, we get an explicit reduced formula. We start with the special case of pairs of Farey determinant 1 or 2.

3.5.1 Pairs with Farey determinant 1 or 2

Recall that to each rational x, we associate the disk $[x]$, passing by $[x]^\sharp$ and $[x]^\flat$. Let us work in the half plane model. The union of all $[x]$ for $x \in \mathbb{P}^1(\mathbb{Q})$ is still denoted by $\mathcal{Q}$.

Using Definition 3.4.1 we can then speak about the inner and outer homothety centers $i([x],[y])$ and $e([x],[y])$ of two q-rationals, where $x, y \in \mathbb{Q}$, $x \neq y$. We extend to the case of $y = \infty$ by $i([x],[\infty]) = [x]^\sharp$ and $e([x],[\infty]) = [x]^\flat$.

Recall also that by Corollary 3.3.4, if two rational numbers x and y have Farey determinant 1 or 2, then the inversion with respect to the geodesic between $[x]$ and $[y]$ is a symmetry of $\mathcal{Q}$. This fact, combined with Proposition 3.4.19, will imply that inner and outer homothety centers of $[x]$ and $[y]$ are q-rational numbers. To be more precise, we need the following lemma.

Lemma 3.5.1. *Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers in reduced form with Farey determinant 1 (resp. 2). Then the Farey determinant between the Farey sum and the Farey difference of $\frac{a}{b}$ and $\frac{c}{d}$ is 2 (resp. 1).*

Proof. Let us denote by d_F the Farey determinant between $\frac{a}{b}$ and $\frac{c}{d}$, and suppose without loss of generality that $\frac{c}{d} < \frac{a}{b}$. Note the following relations : $a(b+d) - b(a+c) = d_F$, and $b(a-c) - a(b-d) = d_F$.
Let us suppose first that $d_F = 1$. Then the fractions $\frac{a+c}{b+d}$ and $\frac{a-c}{b-d}$ are reduced, and the Farey determinant between those is given by

$$d_F\left(\frac{a+c}{b+d}, \frac{a-c}{b-d}\right) = |(a+c)(b-d) - (a-c)(b+d)| = 2\,|ad-bc| = 2.$$

Now, suppose that $d_F = 2$. Then $\gcd(a+c, b+d) = \gcd(a-c, b-d) = 2$, and the Farey determinant we are interested in is given by

$$d_F\left(\frac{a+c}{b+d}, \frac{a-c}{b-d}\right) = \left|\frac{a+c}{2}\frac{b-d}{2} - \frac{a-c}{2}\frac{b+d}{2}\right| = \frac{2d_F}{4} = 1.$$

□

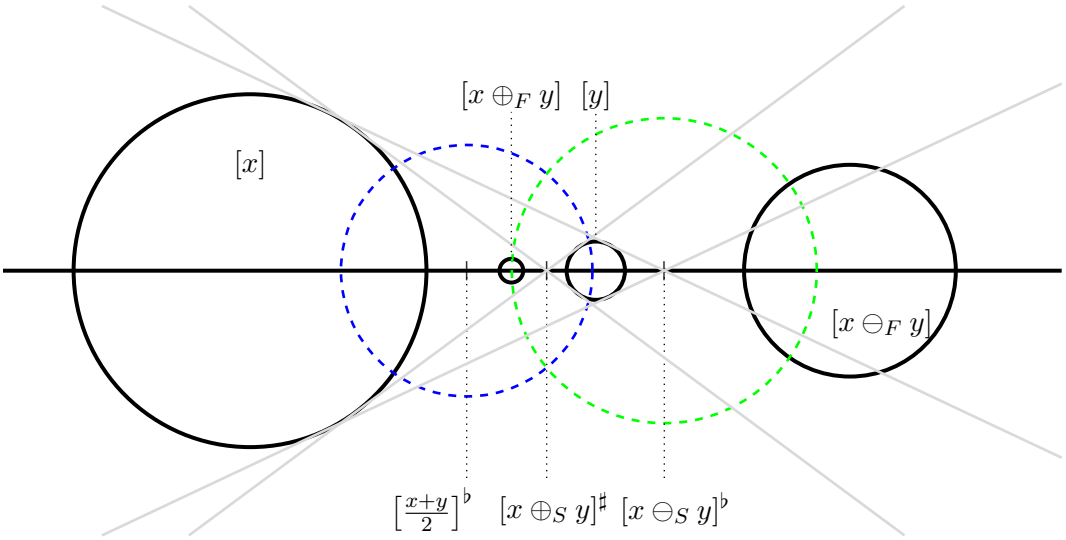


Figure 3.7: A pair $([x], [y])$ with $d_F(x, y) = 1$, and its Farey and Springborn sum and difference. In blue, the geodesic γ between $[x]$ and $[y]$. In green, the geodesic between $[x \oplus_F y]$ and $[x \ominus_F y]$.

We can now relate precisely the homothety centers construction and its algebraic counterpart.

Theorem 3.5.2 ([JPRT26])**.** *Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers with Farey determinant 1 or 2. Then the inner and outer homothety centers of $\left[\frac{a}{b}\right]$ and $\left[\frac{c}{d}\right]$ are q-rational numbers. More precisely,*

$$i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \left[\frac{a}{b} \oplus_S \frac{c}{d}\right]_q^\sharp \quad \text{and} \quad e\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \left[\frac{a}{b} \ominus_S \frac{c}{d}\right]_q^\flat.$$

Proof. Let us consider the pair $\left(\frac{a+c}{b+d}, \frac{a-c}{b-d}\right)$, which also has Farey determinant 1 or 2. The inversion with respect to the geodesic γ between $\left[\frac{a+c}{b+d}\right]$ and $\left[\frac{a-c}{b-d}\right]$ exchanges the circles $\left[\frac{a}{b}\right]$ and $\left[\frac{c}{d}\right]$ (according to Proposition 3.1.18). Thus by Corollary 3.3.4 and Proposition 3.4.19, the center of this inversion, which is $e([\frac{a}{b}], [\frac{c}{d}])$, belongs to $\mathcal{Q}$, and is the left-most point of some q-circle $[z_e]$, as the image of $[\infty]^\sharp$ by a symmetry-preserving operation.
To compute z_e, one can look at the limit $q \to 1$ of the picture. In this case, the geodesic γ becomes the circle orthogonal to the real line with intersection points $\frac{a+c}{b+d}$ and $\frac{a-c}{b-d}$, so the center z_e of this circle is

$$z_e = \frac{1}{2}\left(\frac{a+c}{b+d} + \frac{a-c}{b-d}\right) = \frac{ab-cd}{b^2-d^2}.$$

For the inner homothety center, Proposition 3.4.19 tells that $i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right)$ is the image of $e\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right)$ under the inversion with respect to the geodesic between $\left[\frac{a}{b}\right]$ and $\left[\frac{c}{d}\right]$, so it also belongs to $\mathcal{Q}$ and it is the right-most point of some q-circle $[z_i]$. At $q = 1$, the four points $\frac{a}{b}$, $\frac{c}{d}$, z_i and z_e are harmonic. Together with Proposition 3.4.14, this gives

$$z_i = \frac{ab+cd}{b^2+d^2}.$$

□

3.5.2 General non-reduced expression

We determine an explicit expression for the inner and outer homothetic centers of two q-rationals, seen as circles.

Proposition 3.5.3 ([JPRT26]). *Let $\frac{a}{b}, \frac{c}{d}$ be two rational numbers. Denote by ε_1 (resp. ε_2) the integers associated to a/b (resp. $\frac{c}{d}$), see Definition (1.4.10). Then, we have*

$$i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \frac{q^{\varepsilon_2} A^\sharp B^\flat + q^{\varepsilon_1} C^\flat D^\sharp}{q^{\varepsilon_2} B^\sharp B^\flat + q^{\varepsilon_1} D^\sharp D^\flat}.$$

Similarly, for the outer homothetic center, we have

$$e\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \frac{q^{\varepsilon_2} A^\sharp B^\flat - q^{\varepsilon_1} C^\sharp D^\flat}{q^{\varepsilon_2} B^\sharp B^\flat - q^{\varepsilon_1} D^\sharp D^\flat}.$$

Proof. We combine the explicit formula for homothety centers (Proposition 3.4.15) with the Equation (3.4) for the (Euclidean) diameter of a q-rational. This gives :

$$\begin{aligned} i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) &= \frac{r_2 M_1 + r_1 M_2}{r_1 + r_2} \\ &= \frac{r_2([\frac{a}{b}]^\sharp - r_1) + r_1([\frac{c}{d}]^\flat + r_2)}{r_1 + r_2} \\ &= \frac{\frac{q^{\varepsilon_2}|q-1|}{2D^\sharp D^\flat}\frac{A^\sharp}{B^\sharp} + \frac{q^{\varepsilon_1}|q-1|}{2B^\sharp B^\flat}\frac{C^\flat}{D^\flat}}{\frac{q^{\varepsilon_1}|q-1|}{2B^\sharp B^\flat} + \frac{q^{\varepsilon_2}|q-1|}{2D^\sharp D^\flat}} \\ &= \frac{q^{\varepsilon_2} A^\sharp B^\flat + q^{\varepsilon_1} C^\flat D^\sharp}{q^{\varepsilon_2} B^\sharp B^\flat + q^{\varepsilon_1} D^\sharp D^\flat}. \end{aligned}$$

The same argument holds for the Springborn difference. □

Remark 3.5.4. Using the commutativity of the homothetic centers construction, we also get

$$i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \frac{q^{\varepsilon_2} A^\flat B^\sharp + q^{\varepsilon_1} C^\sharp D^\flat}{q^{\varepsilon_2} B^\sharp B^\flat + q^{\varepsilon_1} D^\sharp D^\flat}.$$

Both formulas give the same result since $A^\sharp B^\flat - A^\flat B^\sharp = q^{\varepsilon_1}(1-q)$ and the same for $\frac{c}{d}$. For the difference, we also get

$$e\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \frac{q^{\varepsilon_2} A^\flat B^\sharp - q^{\varepsilon_1} C^\flat D^\sharp}{q^{\varepsilon_2} B^\sharp B^\flat - q^{\varepsilon_1} D^\sharp D^\flat}.$$

The formulas given above for homothetic centers are q-deformations of the formulas for Springborn operations. As in the classical case, they are not reduced in general. This is a challenging problem to find the reduced forms. We partially solve this problem for regular pairs.

3.5.3 Reduced form

In this subsection, we determine the reduced version of the q-deformed Springborn operations for regular pairs.

Theorem 3.5.5 ([JPRT26]). *Let $(\frac{a}{b}, \frac{c}{d}) \in \mathbb{Q}^2$ be an inner regular pair in the sense of Definition 3.4.6. Denote by $\varepsilon_1 = \varepsilon(\frac{a}{b})$ and $\varepsilon_2 = \varepsilon(\frac{c}{d})$. Then*

$$\gcd(q^{\varepsilon_2} A^\sharp B^\flat + q^{\varepsilon_1} C^\flat D^\sharp, q^{\varepsilon_2} B^\sharp B^\flat + q^{\varepsilon_1} D^\flat D^\sharp, q^{\varepsilon_2} A^\sharp A^\flat + q^{\varepsilon_1} C^\flat C^\sharp) \equiv d_F^{\sharp\flat} \equiv d_F^{\flat\sharp}. \tag{3.6}$$

Similarly for an outer regular pair, we have

$$\gcd(q^{\varepsilon_2} A^\sharp B^\flat - q^{\varepsilon_1} C^\sharp D^\flat, q^{\varepsilon_2} B^\sharp B^\flat - q^{\varepsilon_1} D^\flat D^\sharp, q^{\varepsilon_2} A^\sharp A^\flat - q^{\varepsilon_1} C^\flat C^\sharp) \equiv d_F^{\sharp\sharp} \equiv d_F^{\flat\flat}. \tag{3.7}$$

Note that a similar statement can't hold for non regular pair, since the specialization $q = 1$ in (3.6) and (3.7) gives the definition of a regular pair.

Example 3.5.6. Let us consider $\left(\frac{a}{b}, \frac{c}{d}\right) = (\frac{1}{3}, \frac{5}{2})$. The Farey determinant of this pair is $d_F = 13$, and we have

$$d_F^{\sharp\flat} = 1 + 2q + 2q^2 + 3q^3 + 2q^4 + 2q^5 + q^6.$$

And $\varepsilon_1 = 2$, $\varepsilon_2 = 3$,

$$\begin{aligned} q^{\varepsilon_2} A^\sharp B^\flat + q^{\varepsilon_1} C^\flat D^\sharp &= q^2 d_F^{\sharp\flat}, \\ q^{\varepsilon_2} B^\sharp B^\flat + q^{\varepsilon_1} D^\flat D^\sharp &= q^2 d_F^{\sharp\flat}, \\ q^{\varepsilon_2} A^\sharp A^\flat - q^{\varepsilon_1} C^\flat C^\sharp &= q^2(1+q) d_F^{\sharp\flat}. \end{aligned}$$

The proof strategy goes as follows: we first prove that the equalities are invariant under transformations by T and S. Second, we check them on the representatives given in Proposition 3.4.11.

Proof. We start by showing invariance of the identities under transformations T and S. By Proposition 1.4.31, the Farey determinants are invariant under the $\mathrm{PSL}_2(\mathbb{Z})$-action (up to a some power of q).
Consider a fraction $\frac{a}{b} \in \mathbb{Q}$. Under T, the parameter $\varepsilon = \varepsilon(\frac{a}{b})$ and the (left or right) q-deformation $\frac{A}{B}$ change into

$$\varepsilon \mapsto \begin{cases} \varepsilon + 1 & \text{if } \frac{a}{b} \geq 0 \\ \varepsilon - 1 & \text{if } \frac{a}{b} < 0. \end{cases} \quad \text{and} \quad \frac{A}{B} \mapsto \begin{cases} \frac{qA+B}{B} & \text{if } \frac{a}{b} > 0 \\ \frac{A+B/q}{B/q} & \text{if } \frac{a}{b} < 0. \end{cases}$$

The second part follows from $B^\sharp(0) = 1$ if $\frac{a}{b} \geq 0$, $B^\sharp(0) = 0$ if $\frac{a}{b} < 0$, $B^\flat(0) = 1$ if $\frac{a}{b} > 0$ and $B^\flat(0) = 0$ if $\frac{a}{b} \leq 0$. This also shows that for $\frac{a}{b} = 0$, we are in the first case for the right version, and in the second case for the left version.
With these transformation rules, we prove that the greatest common divisors in the left hand side of Equations (3.6) (3.7) do not change, modulo some power of q. For this, let us denote by t_1, t_2 and t_3 the three terms in the gcds. More precisely, given a pair $(\frac{a}{b}, \frac{c}{d})$, we put:

$$t_1\left(\frac{a}{b}, \frac{c}{d}\right) = q^{\varepsilon_2} A^\sharp B^\flat \pm q^{\varepsilon_1} C^\flat D^\sharp \ , \ t_2\left(\frac{a}{b}, \frac{c}{d}\right) = q^{\varepsilon_2} B^\sharp B^\flat \pm q^{\varepsilon_1} D^\flat D^\sharp \ ,$$

$$t_3\left(\frac{a}{b}, \frac{c}{d}\right) = q^{\varepsilon_2} A^\sharp A^\flat \pm q^{\varepsilon_1} C^\flat C^\sharp.$$

First case: $\frac{c}{d} > 0$. We focus on the case when both $\frac{a}{b}$ and $\frac{c}{d}$ are positive, the negative case works exactly the same.

$$\begin{aligned} t_1\left(T\cdot\frac{a}{b}, T\cdot\frac{c}{d}\right) &= q^{\varepsilon_2+1}(qA^\sharp + B^\sharp)B^\flat \pm q^{\varepsilon_1+1}(qC^\flat + D^\flat)D^\sharp \\ &= q^2 t_1\left(\frac{a}{b},\frac{c}{d}\right) + qt_2\left(\frac{a}{b},\frac{c}{d}\right). \end{aligned}$$

The second term t_2 changes by a factor q under T. The third one changes by

$$\begin{aligned} t_3\left(T\cdot\frac{a}{b}, T\cdot\frac{c}{d}\right) &= q^{\varepsilon_2+1}(qA^\sharp + B^\sharp)(qA^\flat + B^\flat) \pm q^{\varepsilon_1+1}(qC^\sharp + D^\sharp)(qC^\flat + D^\flat) \\ &= q^3 t_3\left(\frac{a}{b},\frac{c}{d}\right) + qt_2\left(\frac{a}{b},\frac{c}{d}\right) + q^2 t_1\left(\frac{a}{b},\frac{c}{d}\right) \\ &+ q^2(q^{\varepsilon_2}A^\flat B^\sharp \pm C^\sharp D^\flat) \end{aligned}$$

By the symmetry between $\sharp$ and $\flat$, see Remark 3.5.4, $t_1\left(\frac{a}{b},\frac{c}{d}\right) = q^{\varepsilon_2}A^\flat B^\sharp \pm C^\sharp D^\flat$, and then

$$\begin{aligned} &\gcd\left(t_1\left(T\cdot\frac{a}{b}, T\cdot\frac{c}{d}\right), t_2\left(T\cdot\frac{a}{b}, T\cdot\frac{c}{d}\right), t_3\left(T\cdot\frac{a}{b}, T\cdot\frac{c}{d}\right)\right) \\ &= \gcd(q^2 t_1\left(\frac{a}{b},\frac{c}{d}\right), qt_2\left(\frac{a}{b},\frac{c}{d}\right), q^3 t_3\left(\frac{a}{b},\frac{c}{d}\right)) \\ &\equiv \gcd(t_1\left(\frac{a}{b},\frac{c}{d}\right), t_2\left(\frac{a}{b},\frac{c}{d}\right), t_3\left(\frac{a}{b},\frac{c}{d}\right)). \end{aligned}$$

In the end, the greatest common divisor of the pair changed by T is the same (up to a factor q) as the one for the initial pair.

Second case: $\frac{ac}{bd} < 0$. We focus on the case when $\frac{a}{b} > 0$ and $\frac{c}{d} < 0$, the symmetric case is similar.
The first term for the pair shifted by T is

$$\begin{aligned} t_1\left(T\cdot\frac{a}{b}, T\cdot\frac{c}{d}\right) &= q^{\varepsilon_2-1}(qA^\sharp + B^\sharp)B^\flat \pm q^{\varepsilon_1-1}(C^\flat + q^{-1}D^\flat)q^{-1}D^\sharp \\ &= t_1\left(\frac{a}{b},\frac{c}{d}\right) + q^{-1}t_2\left(\frac{a}{b},\frac{c}{d}\right). \end{aligned}$$

The second term changes by q^{-1} under T. The third one changes by

$$\begin{aligned} t_2\left(T\cdot\frac{a}{b}, T\cdot\frac{c}{d}\right) &= q^{\varepsilon_2-1}(qA^\sharp + B^\sharp)(qA^\flat + B^\flat) \pm q^{\varepsilon_1+1}(C^\sharp + q^{-1}D^\sharp)(C^\flat + q^{-1}D^\flat) \\ &= qt_3\left(\frac{a}{b},\frac{c}{d}\right) + q^{-1}t_2\left(\frac{a}{b},\frac{c}{d}\right) + 2t_1\left(\frac{a}{b},\frac{c}{d}\right). \end{aligned}$$

Therefore the greatest common divisor for the pair shifted by T is the same as the gcd for the initial pair, up to q.

Third case: $\frac{ac}{bd} = 0$. Here we can explicitly compute the transformation using $\frac{a}{b} = 0$.

A similar argument holds for the transformation S. Under S, the parameter $\varepsilon = \varepsilon(\frac{a}{b})$ and the quantization $\frac{A}{B}$ change into

$$\varepsilon \mapsto \begin{cases} \varepsilon + 1 & \text{if } \frac{a}{b} > 0 \\ \varepsilon - 1 & \text{if } \frac{a}{b} \leq 0. \end{cases} \quad \text{and} \quad \frac{A}{B} \mapsto \begin{cases} \frac{-B}{qA} & \text{if } \frac{a}{b} > 0 \\ \frac{-B/q}{A} & \text{if } \frac{a}{b} < 0. \end{cases}$$

The evolution of ε can be deduced from the direct computation of the q-diameter $\ell_q\left(\frac{-b}{a}\right)$ in terms of $\ell_q\left(\frac{a}{b}\right)$, see (3.4).
The same distinction of cases shows invariance of the greatest common divisor.

To finish the proof, we have to check the identities on some representatives of the equivalence classes of regular pairs under the modular group action. Such representatives are given in Proposition 3.4.11.
Consider first the case of inner regular pairs. A representative is $\left(\frac{1}{0}, \frac{k}{n}\right)$, where $n \mid k^2 + 1$. Since $[\frac{1}{0}]^\sharp = \frac{1}{0}$, $[\frac{1}{0}]^\flat = \frac{1}{1-q}$ and $\varepsilon(\infty) = 0$, we get

$$d_F^{\sharp\flat} = N^\flat \quad \text{and} \quad d_F^{\flat\sharp} = N^\sharp - (1-q)K^\sharp.$$

By Corollary 1.4.35, we see that these two Farey determinants are equal, up to some power of q. It remains to compute the greatest common divisor

$$G_1 = \gcd(q^{\varepsilon_2}(q-1) + K^\flat N^\sharp, N^\sharp N^\flat, q^{\varepsilon_2} + K^\sharp K^\flat).$$

Since $q^{\varepsilon_2}(1-q) = K^\sharp N^\flat - K^\flat N^\sharp$, the first term of G_1 equals $K^\sharp N^\flat$. The third term of G_1 equals

$$\begin{aligned} q^{\varepsilon_2} + K^\sharp K^\flat &= \frac{K^\sharp N^\flat - K^\flat N^\sharp}{1-q} + K^\sharp K^\flat \\ &= \frac{K^\flat((1-q)K^\sharp - N^\sharp) + K^\sharp N\flat}{1-q} \\ &= \frac{-q^\alpha K^\flat N^\flat + K^\sharp N\flat}{1-q}, \end{aligned}$$

where we used Corollary 1.4.35 in the last line. Since $N^\flat(1) \neq 0$, we have $\gcd(N^\flat, q-1) = 1$. Therefore, we see that $N^\flat$ divides G_1. Since $\gcd(K^\sharp, N^\sharp) = 1$, we finally get that

$$G_1 \equiv N^\flat = q^\alpha d_F^{\sharp\flat}$$

Consider now the case of outer regular pairs. A representative is $\left(\frac{1}{0}, \frac{k}{n}\right)$, where $n \mid k^2 - 1$. We then get

$$d_F^{\sharp\sharp} = N^\sharp \quad \text{and} \quad d_F^{\flat\flat} = \frac{N^\flat - (1-q)K^\flat}{q^2 - q + 1}.$$

By Corollary 1.4.35, we see that $d_F^{\sharp\sharp} = d_F^{\flat\flat}$, up to some power of q. It remains to compute the greatest common divisor

$$G_2 = \gcd(q^{\varepsilon_2}(1-q) - K^\sharp N^\flat, N^\sharp N^\flat, q^{\varepsilon_2} - K^\sharp K^\flat).$$

Again, the first term of G_2 equals $-K^\flat N^\sharp$ and the third term equals

$$q^{\varepsilon_2} - K^\sharp K^\flat = \frac{K^\sharp((q-1)K^\flat + N^\flat) - K^\flat N^\sharp}{1-q} = \frac{q^\alpha(q^2-q+1)K^\sharp N^\sharp - K^\flat N^\sharp}{1-q},$$

where we used Corollary 1.4.35. Since $N^\sharp(1) \neq 0$, we have $\gcd(N^\sharp, q-1) = 1$, hence $N^\sharp$ divides G_2. Since $\gcd(K^\flat, N^\flat) = 1$, we finally get

$$G_2 = q^\beta N^\sharp \equiv d_F^{\sharp\sharp},$$

which concludes the proof. □

Combining Proposition 3.5.3 with the computation of Theorem 3.5.5, we get

Corollary 3.5.7. *If $\left(\frac{a}{b}, \frac{c}{d}\right)$ is an inner regular pair, then $i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \frac{K}{N}$, with*

$$K \equiv \frac{q^{\varepsilon_2} A^\sharp B^\flat + q^{\varepsilon_1} C^\flat D^\sharp}{d_F^{\sharp\flat}} \quad \text{and} \quad N \equiv \frac{q^{\varepsilon_2} B^\sharp B^\flat + q^{\varepsilon_1} D^\flat D^\sharp}{d_F^{\sharp\flat}}.$$

A similar formula holds for an outer regular pair.

Combining Theorem 3.5.5 and Proposition 1.4.32, we also get:

Corollary 3.5.8. *If $\left(\frac{a}{b}, \frac{c}{d}\right)$ is an inner regular pair, then its q-Farey determinants $d_F^{\sharp\flat}$ and $d_F^{\flat\sharp}$ are palindromic.*
If the pair is outer-regular, the q-Farey determinants $d_F^{\sharp\sharp}$ and $d_F^{\flat\flat}$ are palindromic.

3.5.4 Characterisation of regular pairs

The following characterization of regular pairs will allow us to generalize Theorem 3.5.2.

Theorem 3.5.9 ([JPRT26])**.** *Let $(\frac{a}{b}, \frac{c}{d}) \in \mathbb{Q}^2$ and denote by d_F their Farey determinant. The following are equivalent:*

(1) the pair is outer regular,

(2) $\gcd(a+c, b+d)\gcd(a-c, b-d) \in \{d_F, 2d_F\}$,

(3) $d_F\left(\frac{a+c}{b+d}, \frac{a-c}{b-d}\right) \in \{1, 2\}$,

(4) there is an orientation-reversing involution in $\mathrm{PGL}_2(\mathbb{Z})$ *exchanging $\frac{a}{b}$ and $\frac{c}{d}$.*

Similarly for inner-regularity, we have the following equivalences, where we work in the quadratic imaginary field $\mathbb{Q}[i]$:

(1') the pair is inner regular,

(2') $\gcd(a+ic, b+id)\gcd(a-ic, b-id) = d_F$,

(3') $d_F\left(\frac{a+ic}{b+id}, \frac{a-ic}{b-id}\right) = 2$,

(4') there is an orientation-preserving involution in $\mathrm{PGL}_2(\mathbb{Z})$ *exchanging* $\frac{a}{b}$ *and* $\frac{c}{d}$.

The proof idea is to leverage the $\mathrm{PSL}_2(\mathbb{Z})$-symmetry to reduce to a representative, and to use Proposition 3.1.17 about involutive elements in $\mathrm{PGL}_2(\mathbb{Z})$.

Proof. (2) $\Leftrightarrow$ (3) *and* (2′) $\Leftrightarrow$ (3′). The equivalence between (2) and (3) is easy, since

$$d_F\left(\frac{a+c}{b+d}, \frac{a-c}{b-d}\right) = \frac{2d_F}{\gcd(a+c, b+d)\gcd(a-c, b-d)}.$$

A similar formula gives the equivalence between (2') and (3').

(2) $\Rightarrow$ (1) *and* (2′) $\Rightarrow$ (1′). To prove that (2) implies (1), we know from Lemma 3.4.8 that it is sufficient to show that d_F divides a^2-c^2, b^2-d^2 and $ab-cd$. By (2) we have that d_F divides $\gcd(a+c, b+d)\gcd(a-c, b-d)$, so

$$d_F \mid a^2-c^2 \text{ and } d_F \mid b^2-d^2.$$

Finally, from the identity

$$(ab-cd)^2 - d_F^2 = (a^2-c^2)(b^2-d^2)$$

we also see that $d_F \mid ab-cd$. A similar argument shows that (2') implies (1').

(1) $\Rightarrow$ (2). To prove that (1) implies (2), notice that both statements are invariant under the $\mathrm{PSL}_2(\mathbb{Z})$-action. Indeed S exchanges a with b and c with d (the sign does not matter since it is a unit), and T maps a to $a+b$ and c to $c+d$, while keeping b and d fixed. These operations do not change the greatest common divisors.
Hence we can reduce to the standard pair $\left(\frac{1}{0}, \frac{k}{n}\right)$. Condition (1) gives that n divides k^2-1, and we have to prove that

$$\gcd(k+1, n)\gcd(k-1, n) \in \{n, 2n\}.$$

If k is even, then $k+1$ and $k-1$ are coprime, so

$$\gcd(k+1, n)\gcd(k-1, n) = \gcd(k^2-1, n) = n.$$

If k and n are odd, we have $\gcd(k+1, n) = \gcd(\frac{k+1}{2}, n)$ and $\frac{k+1}{2}$ is coprime with $\frac{k-1}{2}$. Then

$$\gcd(k+1, n)\gcd(k-1, n) = \gcd(\tfrac{k^2-1}{4}, n) = n.$$

Finally, if k is odd and n is even, we have

$$\begin{aligned}\gcd(k+1, n)\gcd(k-1, n) &= 4\gcd(\tfrac{k+1}{2}, \tfrac{n}{2})\gcd(\tfrac{k-1}{2}, \tfrac{n}{2}) \\ &= 4\gcd(\tfrac{k^2-1}{4}, \tfrac{n}{2}),\end{aligned}$$

which is either n or $2n$. This finishes the proof for outer regularity.

(1′) $\Rightarrow$ (2′). We can again use the $\mathrm{PSL}_2(\mathbb{Z})$-symmetry to reduce to the standard pair $\left(\frac{1}{0}, \frac{k}{n}\right)$. Condition (1') then gives $n \mid k^2+1$, and we have to prove that

$$\gcd(k+i, n)\gcd(k-i, n) = n.$$

We know that $\gcd(k+i, k-i) = \gcd(k+i, 2) \in \{1, 1+i, 2\}$. The value 2 is not possible since 2 does not divide $k+i$.
If $k+i$ and $k-i$ are coprime, or if n and $1+i$ are coprime, we are done, in the same manner as we did before. The only remaining case is when $\gcd(k+i, k-i) = 1+i$ and $\gcd(1+i, n) = 1+i$. Then n is even and we get

$$\begin{aligned}\gcd(k+i, n)\gcd(k-i, n) &= 2\gcd\left(\tfrac{k+i}{1+i}, \tfrac{n}{1+i}\right)\gcd\left(\tfrac{k-i}{1-i}, \tfrac{n}{1-i}\right)\\ &= 2\gcd\left(\tfrac{k+i}{1+i}, \tfrac{n}{2}\right)\gcd\left(\tfrac{k+i}{1+i}, \tfrac{n}{2}\right)\\ &= 2\gcd\left(\tfrac{k^2+1}{2}, \tfrac{n}{2}\right)\\ &= n.\end{aligned}$$

(3) $\Leftrightarrow$ (4). From Proposition 3.1.17, we know that the inversion in the geodesic with endpoints $\frac{a+c}{b+d}$ and $\frac{a-c}{b-d}$ is an orientation-reversing involution in $\mathrm{PGL}_2(\mathbb{Z})$. Since $(\frac{a}{b}, \frac{c}{d}, \frac{a+c}{b+d}, \frac{a-c}{b-d})$ are harmonic, this inversion exchanges $\frac{a}{b}$ and $\frac{c}{d}$. The converse follows from Proposition 3.1.18.

(3′) $\Leftrightarrow$ (4′). Similarly, again from Proposition 3.1.17, we know that the rotation of angle π with center $\frac{a+ic}{b+id}$ (or $\frac{a-ic}{b-id}$ if the latter is in the lower half-plane) is an orientation-preserving involution in $\mathrm{PGL}_2(\mathbb{Z})$. Since $(\frac{a}{b}, \frac{c}{d}, \frac{a+ic}{b+id}, \frac{a-ic}{b-id})$ are harmonic (by a direct computation), these four points are cocyclic. Moreover, since $\frac{a-ic}{b-id}$ is the complex conjugate of $\frac{a+ic}{b+id}$, the corresponding circle is symmetric with respect to the real axis. In other words, the hyperbolic geodesic with endpoints $\frac{a}{b}$ and $\frac{c}{d}$ passes through $\frac{a+ic}{b+id}$, which then implies that involution exchanges $\frac{a}{b}$ and $\frac{c}{d}$. The converse follows from Proposition 3.1.18. □

3.5.5 Springborn operations for regular pairs

We can now generalize Theorem 3.5.2 to the case of regular pairs.

Theorem 3.5.10 ([JPRT26]). *Let $(\frac{a}{b}, \frac{c}{d}) \in \mathbb{Q}^2$. If the pair is inner regular, then*

$$\left[\frac{a}{b} \oplus_S \frac{c}{d}\right]^\sharp = i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right).$$

If the pair is outer regular, then

$$\left[\frac{a}{b} \ominus_S \frac{c}{d}\right]^\flat = e\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right).$$

The proof is a combination of the characterization of regular pairs from Theorem 3.5.9, and the symmetries of the q-Farey tessellation $\mathcal{Q}$.

Proof. Consider an outer regular pair $(\frac{a}{b}, \frac{c}{d})$. From point (4) of Theorem 3.5.9, we know that there is an inversion M in $\mathrm{PGL}_2(\mathbb{Z})$ exchanging $\frac{a}{b}$ with $\frac{c}{d}$. So its q-deformation M_q is a symmetry of $\mathcal{Q}$. By Proposition 3.4.19, we then get that

$$e\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = M_q(\infty).$$

Since IM_q is a symmetry of $\mathcal{Q}$ and $\infty = [\infty]_q^\sharp$, we see that $M_q(\infty)$ is the left-most point of some circle belonging to $\mathcal{Q}$ (orientation-reversing isometries exchange left and right

q-deformations). The label of this circle can be deduced by considering the limit $q \to 1$, exactly as in the proof of Theorem 3.5.2.
The exact same argument, using point $(4')$ of Theorem 3.5.9, works for an inner regular pair. The only difference is that orientation-preserving symmetries of $\mathcal{Q}$ preserve right q-rationals, so that the inner homothety center is the right-most point of some circle belonging to $\mathcal{Q}$. □

Combining Theorem 3.5.10 with Corollary 3.5.7, we get explicit expressions for the q-rationals under Springborn operations. Apart from the $\mathrm{PSL}_2(\mathbb{Z})$-structure of q-rationals (notably expressed via Farey sum and differences), this is the first instance of explicit formulas of this kind.

Corollary 3.5.11. *Let $(\frac{a}{b}, \frac{c}{d})$ be an inner regular pair. Put $\frac{k}{n} = \frac{a}{b} \oplus_S \frac{c}{d}$, i.e. $k = \frac{ab+cd}{ad-bc}$ and $n = \frac{b^2+d^2}{ad-bc}$. Then*

$$K^\sharp \equiv \frac{q^{\varepsilon_2} A^\sharp B^\flat + q^{\varepsilon_1} C^\flat D^\sharp}{d_F^{\sharp\flat}} \tag{3.8}$$

$$N^\sharp \equiv \frac{q^{\varepsilon_2} B^\sharp B^\flat + q^{\varepsilon_1} D^\flat D^\sharp}{d_F^{\sharp\flat}}. \tag{3.9}$$

A similar formula holds for an outer regular pair.

Using these formulas, we can compute q-versions of midpoints between two rationals of Farey determinant 1:

Corollary 3.5.12. *Consider two rational numbers $0 < \frac{a}{b} < \frac{c}{d}$ of Farey determinant 1. Then the arithmetic mean of $\frac{a}{b}$ and $\frac{c}{d}$ has left q-deformation*

$$\left[\frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right)\right]^\flat = \frac{A^\sharp D^\flat + q^\varepsilon C^\sharp B^\flat}{B^\sharp D^\flat + q^\varepsilon B^\flat D^\sharp},$$

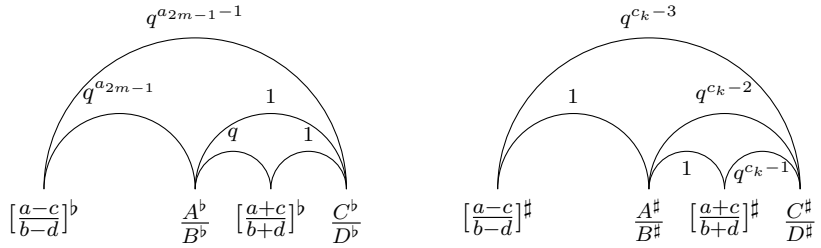
where $\varepsilon = \left|\varepsilon(\frac{a}{b}) - \varepsilon(\frac{c}{d})\right|$ is the difference between the integers associated to $\frac{a}{b}$ and $\frac{c}{d}$ as in (3.3).
A formula for the right q-deformation of the arithmetic mean can be deduced, using the twisted action $[x]_q^\sharp = I_q \cdot [x]_{q^{-1}}^\flat$, see Equation (1.16).

Proof. By Lemma 3.4.13 and Theorem 3.5.10, we have

$$\left[\frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right)\right]^\flat = \left[\left(\frac{a+c}{b+d}\right) \ominus_S \left(\frac{a-c}{b-d}\right)\right]^\flat = e\left(\left[\frac{a}{b} \oplus_F \frac{c}{d}\right], \left[\frac{a}{b} \ominus_F \frac{c}{d}\right]\right).$$

Let $\frac{a+c}{b+d} = [a_1, \cdots, a_{2m}] = [\![c_1, \cdots, c_k]\!]$ be the even positive and negative continued fractions of the Farey sum of $\frac{a}{b}$ and $\frac{c}{d}$. We distinguish cases according to the relationship between $\frac{a}{b}$ and $\frac{c}{d}$.

Case 1 : $\frac{a}{b}$ is a children of $\frac{c}{d}$. The situation in the left and right q-Farey trees is as follows.



Therefore, $\varepsilon(\frac{a+c}{b+d}) = \varepsilon(\frac{a}{b}) + 1$, and $\varepsilon(\frac{a-c}{b-d}) = \varepsilon(\frac{a}{b}) - a_{2m-1}$. And

$$\left[\frac{a+c}{b+d}\right]^\sharp = \frac{A^\sharp + q^{c_k-1}C^\sharp}{B^\sharp + q^{c_k-1}D^\sharp} \quad , \quad \left[\frac{a+c}{b+d}\right]^\flat = \frac{qA^\flat + C^\flat}{qB^\flat + D^\flat},$$
$$\left[\frac{a-c}{b-d}\right]^\sharp = \frac{A^\sharp - q^{c_k-2}C^\sharp}{B^\sharp - q^{c_k-2}D^\sharp} \quad , \quad \left[\frac{a-c}{b-d}\right]^\flat = \frac{q^{-a_{2m-1}}(A^\flat - C^\flat)}{q^{-a_{2m-1}}(B^\flat - D^\flat)}.$$

Denote $\varepsilon_1 = \varepsilon(\frac{a-c}{b-d})$ and $\varepsilon_2 = \varepsilon(\frac{a+c}{b+d})$. Then by Proposition 3.5.3,

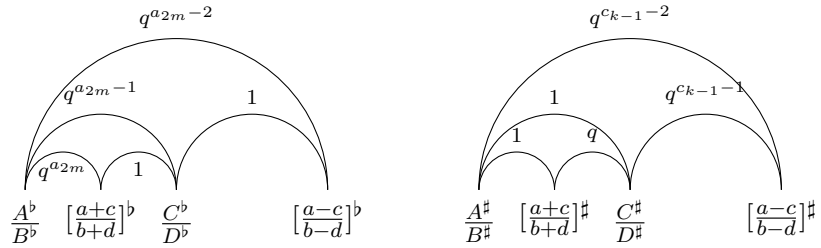
$$e\left(\left[\frac{a}{b} \oplus_F \frac{c}{d}\right], \left[\frac{a}{b} \ominus_F \frac{c}{d}\right]\right)$$
$$= \frac{q^{\varepsilon_1}(A^\sharp + q^{c_k-1}C^\sharp)(qB^\flat + D^\flat) - q^{\varepsilon_2 - a_{2m-1}}(A^\sharp - q^{c_k-2}C^\sharp)(B^\flat - D^\flat)}{q^{\varepsilon_1}(B^\sharp + q^{c_k-1}D^\sharp)(qB^\flat + D^\flat) - q^{\varepsilon_2 - a_{2m-1}}(B^\sharp - q^{c_k-2}D^\sharp)(B^\flat - D^\flat)}$$

Using $\varepsilon_1 + 1 = \varepsilon_2 - a_{2m-1}$ and $\varepsilon_1 + c_k - 1 = \varepsilon_2 + c_k - 2 - a_{2m-1}$, the formula simplifies to

$$e\left(\left[\frac{a}{b} \oplus_F \frac{c}{d}\right], \left[\frac{a}{b} \ominus_F \frac{c}{d}\right]\right) = \frac{q^{\varepsilon_1}(1+q)A^\sharp D^\flat + q^{\varepsilon_1 + c_k - 1}(1+q)C^\sharp B^\flat}{q^{\varepsilon_1}(1+q)B^\sharp D^\flat + q^{\varepsilon_1 + c_k - 1}(1+q)D^\sharp B^\flat},$$

which concludes, since $\varepsilon(\frac{a}{b}) = \varepsilon(\frac{c}{d}) + c_k - 1$.

Case 2 : $\frac{a}{b}$ is the left parent of $\frac{c}{d}$. The situation in the left and right q-Farey trees is then:



Therefore, $\varepsilon(\frac{a+c}{b+d}) = \varepsilon(\frac{c}{d}) + 1$, and $\varepsilon(\frac{a-c}{b-d}) = \varepsilon(\frac{c}{d}) - (c_{k-1} - 1)$. And

$$\left[\frac{a+c}{b+d}\right]^\sharp = \frac{A^\sharp + qC^\sharp}{B^\sharp + qD^\sharp} \quad , \quad \left[\frac{a+c}{b+d}\right]^\flat = \frac{q^{a_{2m}}A^\flat + C^\flat}{q^{a_{2m}}B^\flat + D^\flat},$$
$$\left[\frac{a-c}{b-d}\right]^\sharp = \frac{q^{1-c_{k-1}}(C^\sharp - A^\sharp)}{q^{1-c_{k-1}}(D^\sharp - B^\sharp)} \quad , \quad \left[\frac{a-c}{b-d}\right]^\flat = \frac{C^\flat - q^{a_{2m}-1}A^\flat}{D^\flat - q^{a_{2m}-1}B^\flat}.$$

We still denote by $\varepsilon_1 = \varepsilon(\frac{a-c}{b-d})$ and $\varepsilon_2 = \varepsilon(\frac{a+c}{b+d})$. Then again by Proposition 3.5.3,

$$e\left(\left[\frac{a}{b} \oplus_F \frac{c}{d}\right], \left[\frac{a}{b} \ominus_F \frac{c}{d}\right]\right)$$
$$= \frac{q^{\varepsilon_1}(A^\sharp + qC^\sharp)(q^{a_{2m}}B^\flat + D^\flat) - q^{\varepsilon_2 + 1 - c_{k-1}}(C^\sharp - A^\sharp)(D^\flat - q^{a_{2m}-1}B^\flat)}{q^{\varepsilon_1}(B^\sharp + qD^\sharp)(q^{a_{2m}}B^\flat + D^\flat) - q^{\varepsilon_2 + 1 - c_{k-1}}(D^\sharp - B^\sharp)(D^\flat - q^{a_{2m}-1}B^\flat)}$$

Using $\varepsilon_1 = \varepsilon_2 - c_{k-1}$, the formula simplifies to

$$e\left(\left[\frac{a}{b} \oplus_F \frac{c}{d}\right], \left[\frac{a}{b} \ominus_F \frac{c}{d}\right]\right) = \frac{q^{\varepsilon_1}(1+q)A^\sharp D^\flat + q^{\varepsilon_1+a_{2m}}(1+q)C^\sharp B^\flat}{q^{\varepsilon_1}(1+q)B^\sharp D^\flat + q^{\varepsilon_1+a_{2m}}(1+q)D^\sharp B^\flat},$$

hence the proposition, because $\varepsilon(\frac{c}{d}) = \varepsilon(\frac{a}{b}) + a_{2m}$. □

Remark 3.5.13. It is puzzling that the property described in Theorem 3.5.10 is also valid for some non-regular pairs. An example is given by the pair $\left(\frac{1}{3}, \frac{2}{9}\right)$, which is not regular (inner nor outer), but for which $i([\frac{1}{3}]_q, [\frac{2}{9}]_q) = [\frac{7}{30}]_q^\sharp = [\frac{1}{3} \oplus_S \frac{2}{9}]_q^\sharp$.
Another example is given by the pair $(\frac{2}{7}, \frac{3}{7})$ which is not regular, but for which $e([\frac{2}{7}]_q, [\frac{3}{7}]_q) = [\infty]^\flat = [\frac{2}{7} \ominus_S \frac{3}{7}]^\flat$.
Understanding these exceptional pairs is a challenging task, to which we might come back in the future.

Finding a combinatorial interpretation of identities (3.8) – (3.9) would be an interesting development. In the final section, we address this question in the particular case of Markov fractions.

3.5.6 Application: size of q-rationals

As an application of our main Theorem 3.5.10, we get that the diameters of q-rationals, given by the jump function $\ell_q(\frac{a}{b})$ from Equation (3.4), decrease with the Farey sum in certain conditions.

Theorem 3.5.14 ([JPRT26]). *Let $\frac{a}{b} < \frac{c}{d}$ and suppose that both $\frac{a}{b}, \frac{a+c}{b+d}$ and $\frac{c}{d}, \frac{a+c}{b+d}$ are outer-regular. Denote $\delta = \gcd(a+c, b+d)$. Then*

$$\ell_q\left(\frac{a+c}{b+d}\right) < \ell_q\left(\frac{a}{b}\right) \longleftrightarrow \ell_q\left(\frac{a+c}{b+d}\right) < \ell_q\left(\frac{c}{d}\right) \longleftrightarrow (\delta - 1)b < d.$$

In particular, if $\frac{a}{b}$ and $\frac{c}{d}$ are Farey neighbors, then $\delta = 1$ and the diameters are decreasing with the Farey sum.

Proof. Since T_q scales all diameters by q, we can use it to suppose $0 < \frac{a}{b} < \frac{c}{d}$. The inequality $\ell_q\left(\frac{a+c}{b+d}\right) < \ell_q\left(\frac{a}{b}\right)$ is equivalent to the inequality

$$\left[\frac{a+c}{b+d}\right]^\sharp < e\left(\left[\frac{a}{b}\right], \left[\frac{a+c}{b+d}\right]\right).$$

By Theorem 3.5.10, the outer homothety center can be replaced by the left version of the Springborn difference, so the inequality is equivalent to

$$\left[\frac{a+c}{b+d}\right]^\sharp < \left[\frac{a}{b} \ominus_S \frac{a+c}{b+d}\right]^\flat.$$

By the well-orderedness of q-rational numbers, this is also equivalent to $\frac{a+c}{b+d} < \frac{a}{b} \ominus_S \frac{a+c}{b+d}$. We have

$$\begin{aligned}\frac{a}{b} \ominus_S \frac{a+c}{b+d} - \frac{a+c}{b+d} &= \frac{\delta^2 ab - (a+c)(b+d)}{\delta^2 b^2 - (b+d)^2} - \frac{a+c}{b+d} \\ &= \frac{\delta^2 abd - \delta^2 b^2 c}{(b+d)(\delta^2 b^2 - (b+d)^2)} \\ &= \frac{\delta^2 b}{b+d} \frac{ad-bc}{\delta^2 b^2 - (b+d)^2}\end{aligned}$$

Since $0 < \frac{a}{b} < \frac{c}{d}$, the sign of this is the sign of $(b+d)^2 - \delta^2 b^2$, which is the sign of $(b+d) - \delta b$. In the end, the diameter of $\left[\frac{a+c}{b+d}\right]$ is lower than the diameter of $\left[\frac{a}{b}\right]$ if and only if $d > (\delta - 1)b$.
To compare the diameters of $\left[\frac{a+c}{b+d}\right]$ and $\left[\frac{c}{d}\right]$, the arguments are similar, and in the end the inequality $\ell_q\left(\frac{a+c}{b+d}\right) < \ell_q\left(\frac{c}{d}\right)$ is also equivalent to $d > (\delta - 1)b$. □

The geometric point of view plays a crucial role in the proof of this result. As a comparison, we give an algebraic proof in the very particular case of two Farey neighbours in the positive part of the Farey graph.

Proposition 3.5.15 ([JPRT26]). *Let $0 < \frac{a}{b} < \frac{c}{d} < \infty$ be two rational numbers and suppose they are Farey neighbours, i.e. $|bc - ad| = 1$. Then*

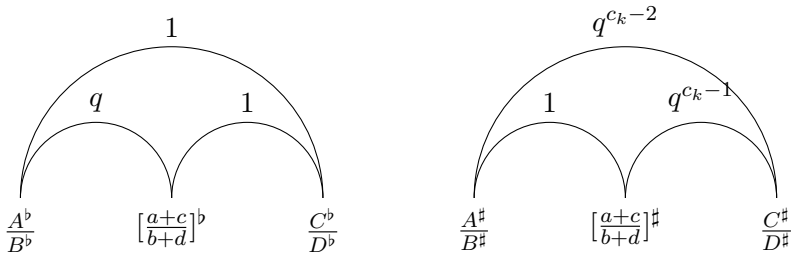
$$\ell_q\left(\frac{a+c}{b+d}\right) < \min\left(\ell_q\left(\frac{a}{b}\right), \ell_q\left(\frac{c}{d}\right)\right).$$

Proof. Consider $\frac{a+c}{b+d} = [a_1, \cdots, a_{2m}] = [\![c_1, \cdots, c_k]\!]$, the even positive and the negative continued fractions.
By Corollary 1.1.18, $\frac{a}{b} = [a_1, \cdots, a_{2m-1}]$ and $\frac{c}{d} = [\![c_1, \cdots, c_{k-1}]\!]$. We distinguish two cases, according to whether $\frac{a}{b}$ is a child of $\frac{c}{d}$ or the contrary.

Case 1 : $\frac{a}{b}$ is a child of $\frac{c}{d}$. This means that $a_{2m} = 1$, and

$$\varepsilon(\tfrac{a}{b}) = \varepsilon(\tfrac{a+c}{b+d}) - 1 \ , \ \varepsilon(\tfrac{c}{d}) = \varepsilon(\tfrac{a+c}{b+d}) - (c_k - 1).$$

The weights needed to compute the left and right q-Farey sums of $\frac{a}{b}$ and $\frac{c}{d}$ are as follows:



Thus the denominator of $\ell_q(\frac{a+c}{b+d})$ is $(B^\sharp + q^{c_k-1}D^\sharp)(qB^\flat + D^\flat)$. By the positivity of q-rationals and because $0 < q$, the real numbers $B^\sharp$, $B^\flat$, $D^\sharp$ and $D^\flat$ are positive. Then

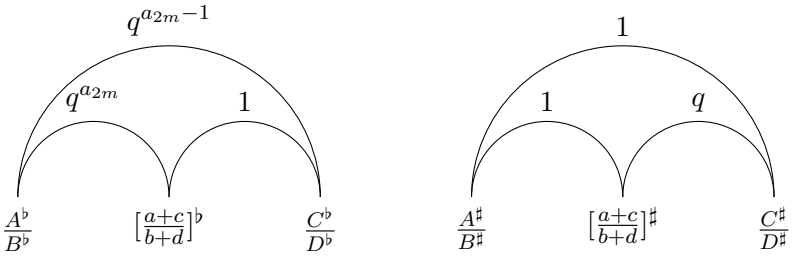
$$\ell_q\left(\frac{a+c}{b+d}\right) < |1-q|\,\frac{q^{\varepsilon(\frac{a+c}{b+d})}}{qB^\sharp B^\flat} = \ell_q\left(\frac{a}{b}\right)$$

$$\ell_q\left(\frac{a+c}{b+d}\right) < |1-q|\,\frac{q^{\varepsilon(\frac{a+c}{b+d})}}{q^{c_k-1}D^\sharp D^\flat} = \ell_q\left(\frac{c}{d}\right).$$

Case 2 : $\frac{a}{b}$ is the left parent of $\frac{c}{d}$. This means that $c_k = 2$, and

$$\varepsilon(\tfrac{a}{b}) = \varepsilon(\tfrac{a+c}{b+d}) - a_{2m}\ ,\ \varepsilon(\tfrac{c}{d}) = \varepsilon(\tfrac{a+c}{b+d}) - 1.$$

The weights needed to compute the left and right q-Farey sums of $\frac{a}{b}$ and $\frac{c}{d}$ are as follows:



Thus the denominator of $\ell_q(\frac{a+c}{b+d})$ is $(B^\sharp + qD^\sharp)(q^{a_{2m}}B^\flat + D^\flat)$. Then

$$\ell_q\left(\frac{a+c}{b+d}\right) < |1-q|\,\frac{q^{\varepsilon(\frac{a+c}{b+d})}}{q^{a_{2m}}B^\sharp B^\flat} = \ell_q\left(\frac{a}{b}\right)$$

$$\ell_q\left(\frac{a+c}{b+d}\right) < |1-q|\,\frac{q^{\varepsilon(\frac{a+c}{b+d})}}{qD^\sharp D^\flat} = \ell_q\left(\frac{c}{d}\right).$$

This concludes the proof. □

Remark 3.5.16. It is not true that the diameter is decreasing with the depth in the Farey graph, even for rationals in $[0,1]$: $\frac{1}{6}$ is deeper in the Farey graph than $\frac{3}{8}$, but for q close to 1 we have

$$\ell_q\left(\frac{1}{6}\right) \underset{q\to 1}{\sim} \frac{|1-q|}{36} > \frac{|1-q|}{64} \underset{q\to 1}{\sim} \ell_q\left(\frac{3}{8}\right).$$

It is not true neither that the diameter decreases with increasing denominator: for q close to 0, we have

$$\ell_q\left(\frac{1}{6}\right) \underset{q\to 0}{\sim} q^5 < q^4 \underset{q\to 0}{\sim} \ell_q\left(\frac{3}{8}\right).$$

As a consequence of the decreasing diameter of q-rationals, we can show that all q-rationals lie in a certain cone:

Corollary 3.5.17. *Let K_q be the cone between the common outer tangents to $[0]$ and $[1]$, whose upper boundary has equation*

$$b = \frac{q-1}{2q\sqrt{q-1+q^{-1}}}((1-q)a - 1).$$

Then $[x] \subset K_q$ for all $x \in \mathbb{R}$.

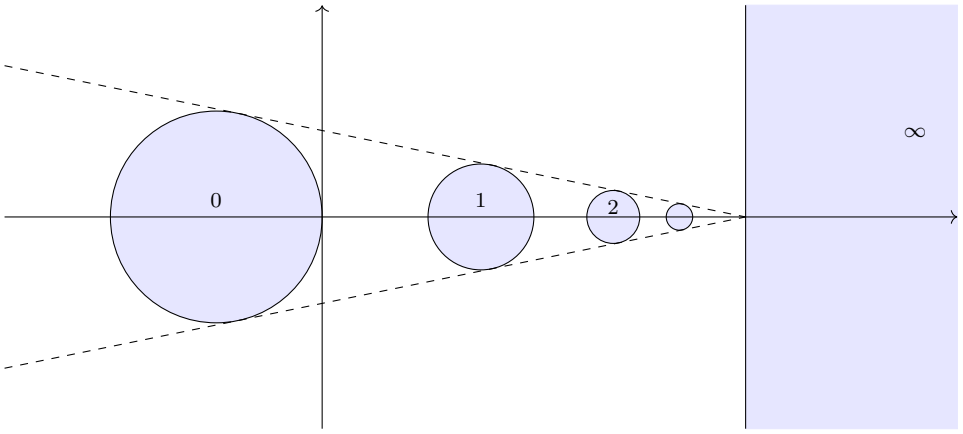


Figure 3.8: The cone K_q containing all disks $[x]$ for $x \in \mathbb{R}$. In this picture, $q = 0.5$.

3.6 Application to Markov fractions

We come to the last q-deformation of Markov numbers studied in this document. Markov fractions were defined by Springborn in [Spr24], they form inner regular pairs so they satisfy Theorem 3.5.10. We show that the q-deformations of rational Markov triples are solutions of a q-deformed Markov equation. We give an additional combinatorial interpretation of Theorem 3.5.10 in this context.

3.6.1 Definition : iterating the Springborn addition

Markov fractions are defined in a recursive tree, starting with the inital two rationals 0 and $\frac{1}{2}$, and iterating the Springborn addition. A given Markov fraction x_1 has thus two parents x_0 and x_2, and the triple (x_0, x_1, x_2) with $x_1 = x_0 \oplus_S x_2$ is called a rational Markov triple. In the rational Markov tree, a vertex represents a rational Markov triple and each zone is labelled by a Markov fraction.

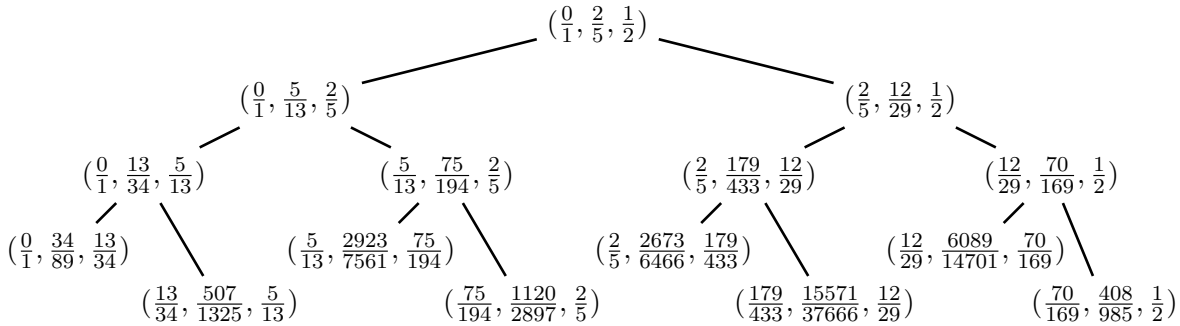


Figure 3.9: First levels of the rational Markov tree

Springborn shows that rational Markov triples satisfy some defining equations. For a

rational Markov triple $\left(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$,

$$\begin{cases} b_0^2 + b_1^2 + b_2^2 = 3b_0b_1b_2 \\ b_0 = b_1a_2 - a_1b_2 \\ b_2 = b_0a_1 - a_0b_1 \end{cases} . \tag{3.10}$$

Recall the theory of Cohn matrices presented in Chapter 2, Definition 2.2.22. The vertices of the Markov tree are parametrized by rational numbers $t \in (0,1)$. For each integer $n \in \mathbb{Z}$, one can define a family of matrices $C^t(n)$, indexed by $(0,1) \cap \mathbb{Q}$, such that $\mathrm{Tr}(C^t(n)) = 3m^t$ is a Markov number (see (2.3)). Markov fractions can be understood in this context as the choice of the parameter $n = 3$.

Proposition 3.6.1 ([JPRT26]). *Let $t \in \mathbb{Q} \cap [0,1]$ and let $\frac{a^t}{m^t}$ be the Markov fraction associated to t. Then the second column of the Cohn matrix $C^t(3)$ is $\begin{pmatrix} m^t \\ -a^t \end{pmatrix}$.*

Proof. We proceed by induction on the 3rd Cohn tree. The first triple is:

$$A(3) = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} \ , \ A(3)B(3) = \begin{pmatrix} 17 & 5 \\ -7 & -2 \end{pmatrix} \ , \ B(3) = \begin{pmatrix} 7 & 2 \\ -4 & -1 \end{pmatrix}.$$

Let (M_0, M_1, M_2) be a Cohn triple, write $M_i = \begin{pmatrix} d_i & b_i \\ -c_i & -a_i \end{pmatrix}$ for $i = 0, 1, 2$, so

$$M_1 = M_0M_2 = \begin{pmatrix} d_1 & d_0b_2 - b_0d_2 = b_1 \\ -c_1 & a_0a_2 - c_0b_2 = -a_1 \end{pmatrix}.$$

Suppose that $(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2})$ is a rational Markov triple. Consider the left child $M_3 = M_0M_1$. Its second column is

$$b_3 = d_0b_1 - b_0a_1 \ , \ -a_3 = -c_0b_1 + a_0a_1.$$

Let us show that $\frac{a_3}{b_3} = \frac{a_0}{b_0} \oplus_S \frac{a_1}{b_1}$. By design of the Cohn tree, b_3 is a Markov number, part of the triple (b_0, b_3, b_1), so $b_3 = \frac{b_0^2+b_1^2}{b_2}$. Since $(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2})$ is a rational Markov triple is satisfies (3.10) so $b_2 = b_0a_1 - a_0b_1$. Then $b_3 = \frac{b_0^2+b_1^2}{a_0b_1-a_1b_0}$ is the denominator of $\frac{a_0}{b_0} \oplus_S \frac{a_1}{b_1}$. We want to show $a_3 = \frac{a_0b_0+a_1b_1}{b_2}$.

$$\begin{aligned} a_0b_0 + a_1b_1 &= a_0(b_1a_2 - a_1b_2) + a_1b_1 \\ &= b_1(c_0b_2 - a_1) - a_0a_1b_2 + a_1b_1 \\ &= b_2(b_1c_0 - a_0a_1) \\ &= b_2a_3. \end{aligned}$$

The right child can be handled with similar arguments. □

Lemma 3.6.2. *Any two rational numbers in a rational Markov triple form an inner regular pair.*

Proof. The initial pair $\left(\frac{0}{1}, \frac{1}{2}\right)$ is inner regular, and $\gcd(1,2) = 1$. Moreover, $1 \mid 0^2 + 1$ and $2 \mid 1^2 + 1$. By iteration conditions determined in Proposition 3.4.23, we have the result. □

We give a q-deformation of the rational Markov tree, replacing each Markov fraction by its right q-version. This gives a new notion of q-deformed Markov numbers, close to the ones studied in [LL22, LLS24, AL25] (but still different). Note that other approaches of q-deformed Markov numbers were explored, based on q-rational numbers as in [Kog20, KO25, EJMGO26], or in a completely different context in [BJO+26].

Theorem 3.6.3 ([JPRT26]). *If* $\left(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$ *is a rational Markov triple, denote by* $\frac{A_i^\sharp}{B_i^\sharp}$ *the right quantized Markov fractions associated with it, and by* $\frac{A_i^\flat}{B_i^\flat}$ *the left one, for* $i \in \{0,1,2\}$. *Then*

$$\begin{cases} B_1^\sharp B_1^\flat + q^{\varepsilon_0+3} B_2^\sharp B_2^\flat + B_0^\flat (B_1^\sharp A_2^\sharp - q^3 A_1^\sharp B_2^\sharp) = [3]_q B_1^\sharp B_2^\sharp B_0^\flat & (r_0) \\ B_0^\sharp \equiv B_1^\sharp A_2^\sharp - A_1^\sharp B_2^\sharp & (r_1) \\ B_0^\flat \equiv B_1^\sharp A_2^\flat - A_1^\sharp B_2^\flat & (r_1^\flat) \\ B_2^\sharp \equiv A_1^\sharp B_0^\sharp - B_1^\sharp A_0^\sharp & (r_2) \\ B_2^\flat \equiv A_1^\flat B_0^\sharp - B_1^\flat A_0^\sharp & (r_2^\flat) \end{cases} \tag{3.11}$$

where ε_0 *was defined in Definition* (1.4.10).

We prove this theorem in Section 3.6.3 below.

Remark 3.6.4. This equation above is a deformation of the classical Markov equation. Note that thanks to relations (3.11), $B_1^\sharp A_2^\sharp - q^3 A_1^\sharp B_2^\sharp$ is a deformation of b_0.

Remark 3.6.5. For a fraction $0 < a/b < 1$, the numerators and denominators of its q-deformation are chosen such that they are coprime polynomials in q, and the denominator has constant coefficient 1. For $\frac{a_0}{b_0} = 0$, one has to choose $A_0^\flat = 1 - q^{-1}$ and $B_0^\flat = 1$ in order to make the equation above work.

3.6.2 Counting in fence posets

The proof of Theorem 3.6.3 relies on a combinatorial interpretation of q-Markov fractions, in terms of fence posets.

Notation 3.6.6. Let $\frac{a}{b}$ be a Markov fraction, different from $\frac{0}{1}$ and $\frac{1}{2}$. By convention, its canonical continued fraction expansion is $\frac{a}{b} = [0, \alpha_1, \cdots, \alpha_n]$, with n even and $\alpha_i \geq 1$ for all i.
The canonical continued fraction expansion of $\frac{0}{1}$ is $[0]$ and the one of $\frac{1}{2}$ is $[0,2]$.

Lemma 3.6.7. *Let* $\left(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$ *be a rational Markov triple. Write the canonical continued fraction expansions of* $\frac{a_0}{b_0}$ *and* $\frac{a_2}{b_2}$ *as*

$$\frac{a_0}{b_0} = [0, \alpha_1, \cdots, \alpha_n] \text{ and } \frac{a_2}{b_2} = [0, \beta_1, \cdots, \beta_m],$$

Then

- $\alpha_1 = \alpha_n = \beta_1 = \beta_m = 2$,
- *the sequences* $(\alpha_1, \cdots, \alpha_n)$ *and* $(\beta_1, \cdots, \beta_m)$ *are palindromic,*

- *the continued fraction expansion of the Springborn sum* $\frac{a_1}{b_1}$ *is given by*

$$\frac{a_1}{b_1} = [0, \alpha_1, \cdots, \alpha_n, 2, 1, \beta_1 - 1, \beta_2, \cdots, \beta_m].$$

Proof. Consider the triangular snake graph model for Markov fractions described by Springborn in [Spr24, Section 5]. This model can be interpreted as a synthesis between two different combinatorial situations :

(i) the snake graph model for Markov numbers, see for example Aigner's book [Aig13];

(ii) the triangulation model for rational numbers, see [MGO19, Section 2].

Indeed, in the Springborn model, each Markov fraction corresponds to a snake graph built with triangles. When gluing pairs of adjacent triangles one get a usual snake graph made of squares (or tiles), which is exactly the snake graph usually associated to the Markov number in the denominator of the Markov fraction. See Figure 3.10 below. On the other hand, in the Springborn triangular snake graph, vertices are labeled with fractions, computed using the Farey summation formula, exactly as in the polygon triangulation model in [MGO19], see Figure 3.11. Combining these two models, we can deduce the continued fractions of Markov fractions.

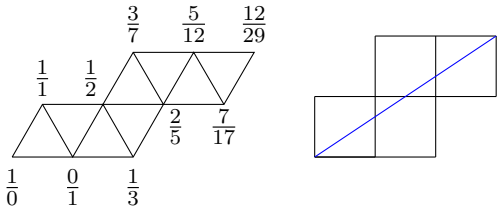


Figure 3.10: Snake graphs associated to the Markov fraction $\frac{12}{29}$.

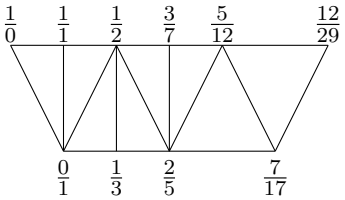


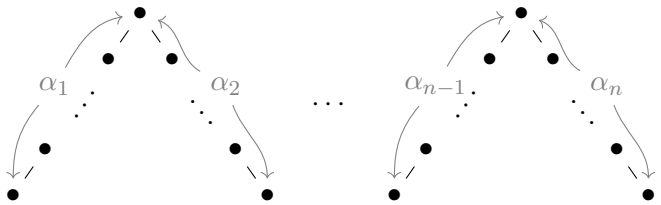
Figure 3.11: Triangulated polygon associated to $\frac{12}{29}$.

More precisely, let μ be a Markov fraction and S_μ its triangular snake graph. This graph has an even number of triangles, so let group them two by two in order to create quadrilateral tiles, starting with the first two adjacent triangles. The continued fraction of μ is obtained by reading S_μ from bottom to top, with the following rules :

- replace the first tile by $[0, 2]$,
- then for each tile, if it is below an other tile, replace these two tiles by $[2, 2]$, and if it is not below an other tile, replace it by $[1, 1]$,
- except for the last tile which is replaced by $[2]$.

Now the usual snake graph model for Markov numbers ensures that the continued fraction we get is palindromic (by palindromicity of Christoffel words, see [Aig13]), and that for two neighbours μ_1 and μ_2 in the Markov tree, the snake graph of there Springborn sum $\mu_1 \oplus \mu_2$ is the concatenation of S_{μ_1} and S_{μ_2}, placing the second one on top of the first one. Hence the formula for the continued fraction. □

Notation 3.6.8. Let $n \in \mathbb{N}^*$ and let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a sequence of n non-negative integers. The corresponding fence poset $F(\alpha)$ is the linear poset with $\alpha_1 + \cdots + \alpha_n + 1$ vertices, and covering relations described by the following Hasse diagram



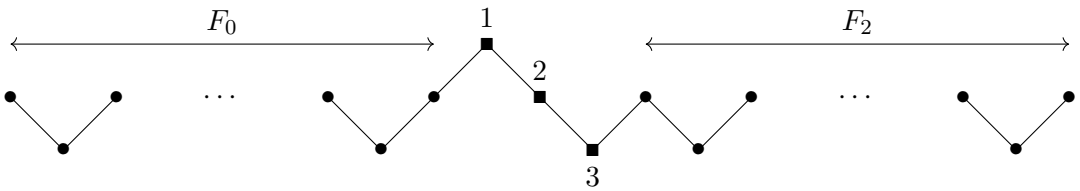
Following [MGO20], we associate to each rational number $0 < x < 1$ a fence poset F_x, such that if $x = [0, \alpha_1, \cdots, \alpha_n]$, then the poset F_x is $F(0, \alpha_1 - 1, \cdots, \alpha_{n-1}, \alpha_n - 1)$.

Now we have tools to state the combinatorial interpretation of our q-Markov numbers.

Lemma 3.6.9. *Let $\left(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$ be a rational Markov triple. Then $B_i^\sharp$ is the generating function of ordered ideals of $F_{\frac{a_i}{b_i}}$, $B_i^\flat$ is the generating function of ordered ideals in $F_{\frac{a_i}{b_i}}$ with the last vertex counting twice, and*

$$\begin{cases} B_1^\sharp = [3]_q B_0^\sharp B_2^\sharp - B_0^\sharp A_2^\sharp + q^3 A_0^\sharp B_2^\sharp \\ B_1^\flat = [3]_q B_0^\sharp B_2^\flat - B_0^\sharp A_2^\flat + q^3 A_0^\sharp B_2^\flat = [3]_q B_0^\flat B_2^\sharp - B_0^\flat A_2^\sharp + q^3 A_0^\flat B_2^\sharp \end{cases} \tag{3.12}$$

Proof. Denote $\frac{a_0}{b_0} = [0, \alpha_1, \cdots, \alpha_n]$ and $\frac{a_2}{b_2} = [0, \beta_1, \cdots, \beta_m]$ the canonical continued fraction expansions. By combinatorial interpretation of q-rationals given in [MGO20] and more precisely in [AL25] for rationals in $(0, 1)$, $B_0^\sharp$ (resp. $B_2^\sharp$) is the generating function of ordered ideals in the fence poset $F_0 := F_{\frac{a_0}{b_0}}$ (resp. $F_2 = F_{\frac{a_2}{b_2}}$) and $A_0^\sharp$ (resp. $A_2^\sharp$) is the generating function of ideals of F_0 (resp. F_2) containing the first α_1 (resp. β_1) vertices. Moreover, by Lemma 3.6.7, the fence poset F_1 is the concatenation of posets F_0 and F_2.



The ideals of this poset F_1 can be split into three groups :

- Group 1 : those which contain vertex 1.
- Group 2 : those which do not contain vertex 1 but contain vertex 2;
- Group 3 : those which do not contain vertex 2 ;

Ideals in the group 1 correspond to the couples of ideals in F_0 containing the last α_n vertices and ideals in F_2, so the generating function of group 1 is $q^3 A_1^\sharp B_2^\sharp$ (because F_0 is symmetric by the palindromicity of the sequence $(\alpha_1, \cdots, \alpha_n)$).

Ideals in the group 2 correspond to the couples of ideals of F_0 and ideals of F_2, so the generating function of group 2 is $q^2 B_0^\sharp B_2^\sharp$.

Ideals in the group 3 correspond to the couples of ideals in F_0 and ideals of the poset $F(1, \beta_1 - 1, \beta_2, \cdots, \beta_m - 1)$, which are counted by the q-numerator of the continued fraction $[2, \beta_1 - 1, \beta_2, \cdots, \beta_m] = ST^2STS(a_2/b_2)$. We have

$$T_q^2 S_q T_q S_q = \begin{pmatrix} -1 & 1+q \\ -1 & 1 \end{pmatrix}$$

so the ideals of the poset $(1, \beta_1 - 1, \beta_2, \cdots, \beta_m - 1)$ are counted by $-A_2^\sharp + (1+q)B_2^\sharp$, and the generating function of the group 3 is $B_0^\sharp(-A_2^\sharp + (1+q)B_2^\sharp)$.

Finally, we get $B_1 = q^3 A_0^\sharp B_2^\sharp + q^2 B_0^\sharp B_2^\sharp - B_0^\sharp A_2^\sharp + (1+q) B_0^\sharp B_2^\sharp$.

On the other hand, the left denominator $B_1^\flat$ is the generating function of F_1 with the final vertex counting twice.
Therefore $B_1^\flat$ is given by the same formula as $B_1^\sharp$ but using the left quantization of a_2/b_2. By palindromicitiy of the sequence defining F_1, the left deformation $B_1^\flat$ is also the generating function of ordered ideals in F_1 with the first vertex counting twice, hence the second formula for $B_1^\flat$. □

3.6.3 Proof of the q-deformed Markov relations

We prove here the following relations for rational Markov triples

$$\begin{cases} B_1^\sharp B_1^\flat + q^{\varepsilon_0+3} B_2^\sharp B_2^\flat + B_0^\flat (B_1^\sharp A_2^\sharp - q^3 A_1^\sharp B_2^\sharp) = [3]_q B_1^\sharp B_2^\sharp B_0^\flat & (r_0) \\ B_0^\sharp \equiv B_1^\sharp A_2^\sharp - A_1^\sharp B_2^\sharp & (r_1) \\ B_0^\flat \equiv B_1^\sharp A_2^\flat - A_1^\sharp B_2^\flat \equiv B_1^\flat A_2^\sharp - A_1^\flat B_2^\sharp & (r_1^\flat) \\ B_2^\sharp \equiv A_1^\sharp B_0^\sharp - B_1^\sharp A_0^\sharp & (r_2) \\ B_2^\flat \equiv A_1^\flat B_0^\sharp - B_1^\flat A_0^\sharp \equiv A_1^\sharp B_0^\flat - B_1^\sharp A_0^\flat & (r_2^\flat) \end{cases}$$

We proceed by induction on the rational Markov tree. It is straightforward to check that relations (3.11) hold for the initial Markov triple $\left(\frac{0}{1}, \frac{2}{5}, \frac{1}{2}\right)$.

Suppose relations (3.11) hold for a rational Markov triple $\left(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$, and consider the child $\frac{a_3}{b_3} = \frac{a_1}{b_1} \oplus_S \frac{a_2}{b_2}$, part of the triple $\left(\frac{a_1}{b_1}, \frac{a_3}{b_3}, \frac{a_2}{b_2}\right)$.

By Theorem 3.5.10,

$$A_3^\sharp \equiv \frac{q^{\varepsilon_2} A_1^\flat B_1^\sharp + q^{\varepsilon_1} A_2^\sharp B_2^\flat}{A_2^\flat B_1^\sharp - A_1^\sharp B_2^\flat} \text{ and } B_3^\sharp \equiv \frac{q^{\varepsilon_2} B_1^\flat B_1^\sharp + q^{\varepsilon_1} B_2^\sharp B_2^\flat}{A_2^\flat B_1^\sharp - A_1^\sharp B_2^\flat}.$$

Because of Lemma 3.6.7, $\varepsilon_1 = \varepsilon_0 + \varepsilon_2 + 3$. Besides, by relation $(r_1^\flat)$, $A_2^\flat B_1^\sharp - A_1^\sharp B_2^\flat \equiv_q B_0^\flat$. Since Markov fractions are in $(0,1)$, their q-deformations must have denominators with

constant coefficient 1, so we can normalize and get

$$A_3^\sharp = \frac{A_1^\flat B_1^\sharp + q^{\varepsilon_0+3} A_2^\sharp B_2^\flat}{B_0^\flat} \text{ and } B_3^\sharp = \frac{B_1^\flat B_1^\sharp + q^{\varepsilon_0+3} B_2^\sharp B_2^\flat}{B_0^\flat}.$$

Let us show relation (r_1) for the child.

$$\begin{aligned} B_3^\sharp A_2^\sharp - A_3^\sharp B_2^\sharp &= \frac{B_1^\flat B_1^\sharp + q^{\varepsilon_0+3} B_2^\sharp B_2^\flat}{B_0^\flat} A_2^\sharp - \frac{A_1^\flat B_1^\sharp + q^{\varepsilon_0+3} A_2^\sharp B_2^\flat}{B_0^\flat} B_2^\sharp \\ &= B_1^\sharp \frac{A_2^\sharp B_1^\flat - B_2^\sharp A_1^\flat}{B_0^\flat} \\ &\equiv_q q^{\varepsilon_2} B_1^\sharp \end{aligned}$$

where the last equality comes from relation $(r_1^\flat)$.
The relation (r_2) for the child follows from similar computations, using the other expression of $A_3^\sharp = \frac{A_1^\sharp B_1^\flat + q^{\varepsilon_0+3} A_2^\flat B_2^\sharp}{B_0^\flat}$ (recall Remark 3.5.4).
First equality of relation $(r_1^\flat)$ for the child is straightforward. The second equality is true up to a power of q by Theorem 3.5.5. Relation $(r_2^\flat)$ is symmetric.
Now relation (r_0) for the child comes from Lemma 3.6.9 applied to the triple $(\frac{a_3}{b_3}, \frac{a_3}{b_3} \oplus_S \frac{a_2}{b_2}, \frac{a_2}{b_2})$, combined with Theorem 3.5.10 :

$$[3]_q B_3^\sharp B_2^\sharp - B_3^\sharp A_2^\sharp + q^3 A_3^\sharp B_2^\sharp = \frac{B_3^\flat B_3^\sharp + q^{\varepsilon_1+3} B_2^\sharp B_2^\flat}{B_1^\flat}.$$

To finish induction, it remains to check the same relations (3.11) for the other child of the triple $\left(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$, with $\frac{a_0}{b_0} \oplus_S \frac{a_1}{b_1}$, and the arguments are symmetric to the previous case. This concludes the proof of Theorem 3.6.3.

3.7 Beyond the real line...

Ultimately, the goal of the project with Olga Paris-Romaskevich and Alexander Thomas is to develop a geometric theory of q-complex numbers. In this perspective, the study of Springborn operations on q-rational numbers is just a first step toward a more general understanding of the underlying rules of the q-deformed numbers.

There already exists a definition of q-complex numbers, proposed by Etingof, extending the quantization map $[\cdot]_q : \mathbb{R} \longrightarrow \mathbb{R}$ to a meromorphic function $\mathbb{C}_+ \longrightarrow \mathbb{C}$ (the variable q is assumed to be a real parameter), by requiring a global equivariance with respect to the modular group. This deformation of complex numbers satisfies nice analytical properties, in particular it can be computed using hypergeometric functions.

The whole interest of the concept of deformation being that one can deform an object in many ways according to what is most relevant in the mathematical context one is interested in, we propose to work on another approach to quantize complex numbers. This approach, initiated by Alexander Thomas, postulates that q-complex numbers are disks. To construct these disks, our starting point will be the deformation of extensions of the modular group acting on the complex plane, combined with a generalization of geometric operations on q-numbers.

Chapter 4

Braids and q-rational numbers in higher dimension

Je tresserai mes vers de verre et de verveine

La Constellation, Les Yeux d'Elsa,
Louis Aragon

The theory of q-rational numbers relies on the deformation of the action of the modular group on the rational line. As shown by Morier-Genoud, Ovsienko and Veselov [MGOV24], this deformed action actually corresponds to the Burau representation of the braid group B_3, via the isomorphism between the modular group and B_3 quotiented by its center. Since B_3 belongs to a family of groups defined for all integer $n \geq 3$, it is tempting to try and generalize the construction of q-rational numbers in higher dimension thanks to the Burau representations of higher braid groups. We investigate in this chapter the case of B_4 acting on the plane.

We begin in Section 4.1 with some generalities about braid groups and the Burau representation, that we define algebraically. Then we focus on the case of the group B_4, studying in Section 4.2 its action on $\mathbb{P}^2(\mathbb{Q})$ given by the integral Burau representation (which we think of the classical case), and deducing from this study in Section 4.3 a deformation of the plane. In Section 4.4, we explore the general case of the action of B_n on $\mathbb{P}^{n-2}(\mathbb{Q})$ via the integral Burau representation.

The major part of this chapter is adapted from [Jou25a].

Main results of Chapter 4. The first main contribution of this chapter is the precise description of the action of B_4 on the rational plane, summarized in

- Theorem 4.2.4 and Corollary 4.2.5, which give an explicit decomposition of $\mathbb{P}^2(\mathbb{Q})$ in orbits,
- Theorem 4.2.10, which computes the stabilizers of the corresponding orbits.

The second important result is the compatibility between our quantization of the plane and the deformation map of Morier-Genoud and Ovsienko, see

- Theorem 4.3.6, stating that the two deformation maps commute.

Finally, the last section is dedicated to establishing a classification of orbits in $\mathbb{P}^{n-2}(\mathbb{Q})$ under the action of B_n for general n, and we get

- Theorem 4.4.3 dealing with the case when n is odd,
- Theorem 4.4.8 for the case when n is even.

Contents

4.1 Burau representation of braid groups

For our purpose, only the algebraic point of view on braid groups will be needed, so we will not present the rich topological aspects of braid theory. We refer to the book [KT08] for a detailed exposition.

4.1.1 Braid groups B_n

Definition 4.1.1. Let $n \in \mathbb{Z}_{\geq 1}$. The *Artin braid group* B_n with n strands is the group presented by

$$B_n := \left\langle \sigma_1, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \;,\; i \in \{1, 2, \cdots, n-2\} \end{array} \right\rangle . \tag{4.1}$$

Remark 4.1.2. By convention, the group B_1 is the trivial group.

Example 4.1.3. For $n = 2$, the Artin braid group with 2 strands is isomorphic to $\mathbb{Z}$, as it has one generator σ_1, and no relation.

Example 4.1.4. For $n = 3$, the Artin braid group with three strands has two generators and only one relation,

$$B_3 := \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle . \tag{4.2}$$

We will need to understand centers of braid groups.

Definition 4.1.5. Let $n \in \mathbb{Z}_{\geq 3}$, and denote

$$\Delta_n = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})\ldots(\sigma_1\sigma_2)\sigma_1.$$

This element $\Delta_n \in B_n$ is called the *Garside element* of the group B_n.

The following proposition is proved for instance in [KT08].

Proposition 4.1.6. *For $n \geq 3$, the center of B_n is the group generated by Δ_n^2.*

$$Z(B_n) = \langle \Delta_n^2 \rangle.$$

The first non trival braid group B_3 will be of interest for us in the sequel. We thus give a well-known description of it, and reproduce the classical proof using our notations of Chapter 1. Recall our preferred generators of $\mathrm{PSL}_2(\mathbb{Z})$:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 4.1.7. *The group B_3 satisfy another presentation, $B_3 \simeq \langle x, y \mid x^2 = y^3 \rangle$. Moreover, the following map is an isomorphism*

$$\begin{array}{rccc} \phi: & B_3/Z(B_3) & \xrightarrow{\simeq} & \mathrm{PSL}_2(\mathbb{Z}) \\ & x & \longmapsto & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ & y & \longmapsto & \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \end{array}. \tag{4.3}$$

Proof. Set $x = \sigma_1\sigma_2\sigma_1 = \Delta_3$ and $y = \sigma_1\sigma_2$ to get this new presentation of B_3. This allows to define a group homorphism $B_3 \longrightarrow \mathrm{PSL}_2(\mathbb{Z})$ mapping x to S and y to ST^{-1}. The kernel of this homorphism is $\langle x^2 \rangle = Z(B_3)$, and hence the isomorphism written in the lemma. □

4.1.2 The Burau representation as a deformation

We present a representation of B_n introduced by Werner Burau in 1935 [Bur35]. It is a n-dimensional representation, first defined algebraically, and later realized in a homological context. It turns out that the Burau representation is reducible, and contains a $(n-1)$-dimensional subrepresentation, called the reduced Burau representation. One of the interest of this reduced Burau representation is its connection with the Alexander polynomial, the first link invariant defined by Alexander in 1928 [Ale28]. In this chapter, we will only work with the reduced version, so for the rest of the document, by Burau representation we mean the reduced one.

To match with the q-analogue context, we use the parameter q instead of the more usual $t = -q$.

Let $n \in \mathbb{Z}_{\geq 2}$. Set $B = \begin{pmatrix} 1 & 0 & 0 \\ -q & q & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Then define the $(n-1) \times (n-1)$-matrices

$$V_1 = \begin{pmatrix} q & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}, V_i = \begin{pmatrix} I_{i-2} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I_{n-i-2} \end{pmatrix}, V_{n-1} = \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & q \end{pmatrix}.$$

for $1 < i < n-1$.

Lemma 4.1.8. *The map $\rho_n : B_n \longrightarrow \mathrm{GL}_{n-1}(\mathbb{Z}[q,q^{-1}])$, mapping σ_i to V_i, is a well-defined group homomorphism.*

Definition 4.1.9. The map ρ_n of the above Lemma is called the *Burau representation* of the braid group B_n.
The *projective Burau representation* is the same homorphism composed with the projectivization $\mathrm{GL}_{n-1}(\mathbb{Z}[q,q^{-1}]) \longrightarrow \mathrm{PGL}_{n-1}(\mathbb{Z}[q,q^{-1}])$.

Remark 4.1.10. The image of the projective Burau representation $\rho_3 : B_3 \longrightarrow \mathrm{PGL}_2(\mathbb{Z}[q,q^{-1}])$ is exactly the q-version of the modular group $\mathrm{PSL}_{2,q}(\mathbb{Z})$ of §1.2.2. Actually, the map ρ_3 establishes the following commutative diagram.

$$\begin{array}{ccc} B_3/Z(B_3) & \xrightarrow{\rho_3} & \mathrm{PSL}_{2,q}(\mathbb{Z}) \\ \wr\downarrow & \circlearrowleft & \nearrow_{[\cdot]_q} \\ \mathrm{PSL}_2(\mathbb{Z}) & & \end{array} \qquad \text{with} \qquad \begin{array}{l} \rho_3(\sigma_1) = T_q \\ \rho_3(\sigma_2) = L_q^{-1} \end{array}$$

It is well-known that the Burau representation of ρ_3 provides a way to realize B_3 as a linear group. We write a proof for completeness.

Proposition 4.1.11. *For $n = 3$, the Burau representation ρ_3 is faithful.*

Proof. Let $\beta \in B_3$ be such that $\rho_3(\beta) = \mathrm{Id} \in \mathrm{GL}_2(\mathbb{Z}[q,q^{-1}])$. By evaluating $q = 1$, we have $\rho_3(\beta)(q=1) = \mathrm{Id} \in \mathrm{PSL}_2(\mathbb{Z})$. Thus by the isomorphism (4.3), we have $\beta \in Z(B_3) = \langle x^2 \rangle$, with $x = \sigma_1\sigma_2\sigma_1$. Let $k \in \mathbb{Z}$ such that $\beta = x^{2k}$.
A direct computation gives $\rho_3(\beta) = (-q^3)^k\,\mathrm{Id}$, and then $\rho_3(\beta) = \mathrm{Id}$ implies that $k = 0$. □

The question of faithfulness for higher Burau representations is a major problem of low-dimensional topology, and was extensively studied [Moo91, LP93]. The question was solved for $n \geq 5$ in 1999 by Bigelow, but the case $n = 4$ is still open at the time this document is written. The faithfulness of ρ_4 is a necessary condition to the Jones Unknot conjecture, one of the most important question of knot invariant theory, stating that the Jones polynomial can detect the unknot.

Theorem 4.1.12 ([Big99])**.** *For $n \geq 5$, the Burau representation ρ_n is not faithful.*

In a different direction, one can also study the behaviour of Burau representation through evaluations of the parameter q. There is a large literature on the subject [Ven14, Sch20, FK09], and recently the relationship between q-rational numbers and ρ_3 was exploited in [MGOV24] to establish the faithfulness of specializations of ρ_3 at complex values.

4.1.3 Integral Burau representation

Definition 4.1.13. Let $z \in \mathbb{C} \setminus \{0\}$. We consider the *evaluation map*

$$\mathrm{ev}_z : \mathrm{PGL}_{n-1}(\mathbb{Z}[q,q^{-1}]) \longrightarrow \mathrm{PGL}_{n-1}(\mathbb{C}),$$

specializing $q = z$.

Definition 4.1.14. For $n \geq 3$, the *integral Burau representation* of the braid group B_n, denoted by ρ_n^{int}, is the composition of the Burau representation of B_n with this evaluation map ev_1.

$$\rho_n^{int} = \mathrm{ev}_1 \circ \rho_n.$$

Notation 4.1.15. The kernel of the integral Burau representation is called the *braid Torelli group*, and denoted $\mathcal{BI}_n := \mathrm{Ker}(\rho_n^{int})$.

Remark 4.1.16. For $n = 3$, the integral Burau representation ρ_3^{int} is the group homorphism $B_3 \longrightarrow \mathrm{PSL}_2(\mathbb{Z})$ we already encountered in (4.3). The braid Torelli group in this case is then the center of the braid group $\mathcal{BI}_3 = Z(B_3)$.

The braid Torelli groups and related objects are well studied in the literature. In 1979, Smythe produced a set of normal generators of $\mathcal{BI}_4$.

Proposition 4.1.17 ([Smy79])**.** *The group $\mathcal{BI}_4$ is normally generated by the three braids τ_1^2, τ_3^2 and Δ_4^2, where*

$$\tau_1 := (\sigma_1\sigma_2\sigma_1)^2 \ , \ \tau_3 = (\sigma_3\sigma_2\sigma_3)^2 \ , \ \Delta_4 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1.$$

More recently, using a topological interpretation of braid groups, Brendle, Margalit and Putman gave a generalization of the result of Smythe for all $n \geq 2$ [BMP15].

4.2 Action of B_4 on the rational plane

The Burau representation of B_3 gives the deformation of the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the rational line. We propose a higher dimensional generalization of this fact, using the Burau representation of B_4 to quantize the rational plane. This section was published in the paper [Jou25a].

One major difference when passing from B_3 to B_4 is the behaviour of the Burau representation at $q = 1$: if it realized an isomorphism between $B_3/Z(B_3)$ and $\mathrm{PSL}_2(\mathbb{Z})$, for $n = 4$ it is not the case anymore.

4.2.1 Description of orbits and stabilizers

From now on, we will work with the projective integral Burau representation of B_4, which is the group homorphism $\rho_4^{int} : B_4 \to \mathrm{PSL}_3(\mathbb{Z})$, defined on generators by

$$\rho_4^{int}(\sigma_1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_4^{int}(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_4^{int}(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}. \tag{4.4}$$

We are interested in the action of B_4 on the rational projective plane, induced by the integral Burau representation.

Notation 4.2.1. Every point $p \in \mathbb{P}^2(\mathbb{Q})$ will be written with homogeneous coordinates $p = [r : s : t]$, normalized so that r, s and t are mutually coprime integers. There are two such triples (r, s, t) for any point p, and they are opposite of each others.

The action $B_4 \curvearrowright \mathbb{P}^2(\mathbb{Q})$ preserves the convention on coordinates. On the generators of B_4, this action is

$$\begin{array}{rccl} \rho_4^{int}(\sigma_1): & [r:s:t] & \mapsto & [r+s:s:t], \\ \rho_4^{int}(\sigma_2): & [r:s:t] & \mapsto & [r:s+t-r:t], \\ \rho_4^{int}(\sigma_3): & [r:s:t] & \mapsto & [r:s:t-s]. \end{array} \tag{4.5}$$

The rest of the paragraph is dedicated to the study of orbits and stabilizers of this action.

Lemma 4.2.2. *(i) For every $[r:s:t] \in \mathbb{P}^2(\mathbb{Q})$, the number $n := \gcd(r-t,s)$ is invariant under the action of B_4.*

(ii) Up to the sign, the number $r \bmod (n) = t \bmod (n)$ is invariant under the action of B_4.

Proof. Part (i). As $\gcd(r+s-t,s) = \gcd(r-t,s+r-t) = \gcd(r-t,s)$, the actions of the generators σ_1, σ_2, σ_3 given by (4.5) do not change the quantity $\gcd(r-t,s)$.

Part (ii). The quantities $r \bmod (n)$ and $t \bmod (n)$ coincide because n divides $r-t$ so $r \equiv t \mod (n)$. Because of the sign change $[r:s:t] = [-r:-s:-t]$, this value is only defined up to the sign. It is straightforward to check that it is invariant under the action of the generators of B_4. □

The above lemma implies that points $[m:n:m]$, with $n \in \mathbb{N}^*$ and m prime to n such that $0 \leq m < n/2$, belong to different orbits under B_4. We are going to prove that they form a system of representatives for this action, using an Euclidean-like algorithm to explore orbits[1].

Remark 4.2.3. One has

$$[m:n:m] = \rho_4^{int}(\sigma_1\sigma_2^2\sigma_3)\left([n-m:n:n-m]\right),$$

so that $[m:n:m]$ and $[n-m:n:n-m]$ belong to the same orbit. Therefore the condition $0 \leq m < n/2$ is indeed necessary to have different orbits.
Note also that when $n \geq 2$, one cannot take $m = 0$ since $[0:n:0] = [0:1:0]$, so that the condition reads $0 < m < n/2$ in this case.

Theorem 4.2.4 ([Jou25a])**.** *Every point $p \in \mathbb{P}^2(\mathbb{Q})$ belong to the orbit of some $[m:n:m]$, with $n, m \in \mathbb{Z}_{\geq 0}$. Moreover, the* braided Euclidean algorithm *below gives in finite steps a braid β such that $\rho_4^{int}(\beta) \cdot p = [m:n:m]$.*

Braided Euclidean algorithm

Input: Start with a point $[r:s:t] \neq [1:0:1]$, with $r, s, t \in \mathbb{Z}$, mutually prime. Assume that $s \geq 0$.

Step 1 of the algorithm: If $s = 0$, then apply σ_2 to replace s by $t-r$, and if necessary change signs to get $s > 0$.

[1] I aknowledge Oleg Karpenkov for the design of the algorithm latex environment.

Write the Euclidean division of r by s and t by s :

$$r = sa_1 + r' \quad t = sc_1 + t'$$

Apply $\sigma_1^{-a_1}\sigma_3^{c_1}$, so that $[r : s : t] \mapsto [r' : s : t'] =: [r_1 : s_1 : t_1]$.

Step 2 of the algorithm: While $r_i - t_i \neq 0$, repeat :
Write the upper Euclidean division of s_i by $(r_i - t_i)$, i.e.

$$s_i = (r_i - t_i)b_{2i} + s' \text{ with } 0 < s' \leq |r_i - t_i|\,.$$

Apply $\sigma_2^{b_{2i}}$ and put $s_{i+1} := s'$.
Write the Euclidean divisions of r_i and t_i by s_{i+1} :

$$r_i = s_{i+1}a_{2i+1} + r' \quad t_i = s_{i+1}c_{2i+1} + t'.$$

Apply $\sigma_1^{-a_{2i+1}}\sigma_3^{c_{2i+1}}$ and put $r_{i+1} := r'$ and $t_{i+1} := t'$.
Termination of the algorithm: If $r_i = t_i$, do not proceed further. Here the algorithm terminates.

Proof. The case of the point $[1 : 0 : 1]$ can be treated separately since it is a fixed point of the action of B_4. That is why the algorithm begins with a point distinct from $[1 : 0 : 1]$. We first prove that the braided Euclidean algorithm terminates, by noticing that thanks to the Euclidean division property, at each step of the algorithm the inequalities below hold:

$$\text{(I) } |r_{i+1} - t_{i+1}| < s_{i+1} < |r_i - t_i| < s_i \text{ and (II) } 0 \leq r_i < s_i, \quad 0 \leq t_i < s_i.$$

Therefore the sequences $(s_i)_i$ and $(|r_i - t_i|)_i$ are strictly decreasing, so the sequence $(|r_i - t_i|)_i$ reaches 0 in a finite number N of steps.

Besides, when $i = N$, we have $r_N = t_N$ so in the end we get a point of the type $[m : n : m]$, with $0 \leq m < n$ by (II).
At each step the algorithm uses the action of one elementary braid σ_i, so one can recover a braid β sending the starting point $[r : s : t]$ to the representative $[m : n : m]$. This braid can be expressed as

$$\beta = \sigma_1^{-a_{2k+1}}\sigma_3^{c_{2k+1}}\sigma_2^{b_{2k}} \;\cdots\; \sigma_1^{-a_3}\sigma_3^{c_3}\sigma_2^{b_2}\sigma_1^{-a_1}\sigma_3^{c_1}\sigma_2^{b_0},$$

where $b_0 = \delta_{s,0}$ and the a_i's and c_i's are uniquely defined by the algorithm. □

Corollary 4.2.5. *(i) Under the B_4-action, the rational projective plane is decomposed into infinitely many orbits as follows*

$$\mathbb{P}^2(\mathbb{Q}) = \{[1 : 0 : 1]\} \sqcup \mathrm{Orb}_{B_4}([0 : 1 : 0]) \sqcup \bigsqcup_{\substack{n\geq 2, 0<m<n/2 \\ \gcd(m,n)=1}} \mathrm{Orb}_{B_4}([m : n : m]).$$

(ii) For every couple (m,n) of coprime integers, the orbit $\mathrm{Orb}_{B_4}([m:n:m])$ is described by

$$\mathrm{Orb}_{B_4}([m:n:m]) = \left\{ [r:s:t] \in \mathbb{P}^2(\mathbb{Q}) \;\middle|\; \begin{cases} \gcd(r-t,s) = n; \\ r,t \equiv \pm m \mod (n). \end{cases} \right\},$$

and

$$\mathrm{Orb}_{B_4}([0:1:0]) = \{[r:s:t] \mid \gcd(r-t,s) = 1\}.$$

Proof. By Theorem 4.2.4, the rational projective plane is a union of orbits of $[m:n:m]$, for coprime integers m, n. According to Lemma 4.2.2, a set of representatives is given by the points $[m:n:m]$ with $n \geq 1$ and m coprime to n, with $0 \leq m < n/2$ (plus the single point $[1:0:1]$). This proves the first part of the Corollary.
The second part is a direct consequence of Lemma 4.2.2. □

Example 4.2.6. Let us apply the braided Euclidean algorithm to $x = [37:30:12]$. As $\gcd(37-12,30) = 5$ and $37 \equiv 2 \mod (5)$, we know that x is in the orbit of $[2:5:2]$.
First step : $37 = 30 * 1 + 7$ so $x \xrightarrow{\sigma_1^{-1}} [7:30:12]$.
Second step : $30 = |7-12| * 5 + 5$ so $[7:30:12] \xrightarrow{\sigma_2^{-5}} [7:5:12]$ and then $[7:5:12] \xrightarrow{\sigma_1^{-1}\sigma_3^2} [2:5:2]$.
Finally the braid $\sigma_1^{-1}\sigma_3^2\sigma_2^{-5}\sigma_1^{-1}$ sends x to the representative of its orbit, $[2:5:2]$.

Remark 4.2.7. The algorithm does not take into account the fact that $[m:n:m]$ is in the same orbit as $[n-m:n:n-m]$. If we really want to have only the representatives of the form $[m:n:m]$ with $m < n/2$ in the end of the algorithm, we can just add $\sigma_1\sigma_2^2\sigma_3$ as a final step in case we reached the wrong representative.

Remark 4.2.8. The braided Euclidean algorithm fits into the framework of Jacobi-Perron type multidimensional continued fraction (MCF) algorithms [Kar21], with defining subsets of $\mathbb{R}^3_+$

$$I_0 = \{(r,s,t) \mid s = \min(r,s,t)\}$$
$$I_1 = \{(r,s,t) \mid t < s \leq r\}\ ,\ I_3 = \{(r,s,t) \mid r < s \leq t\}$$
$$I_2 = \{(r,s,t) \mid t < r < s\}\ ,\ I_2' = \{(r,s,t) \mid r < t < s\}$$

In order to describe stabilizers of the system of representatives, recall the generators of the kernel of ρ_4^{int}, given in Proposition 4.1.17.

Notation 4.2.9. In B_4, consider $\tau_1 = (\sigma_1\sigma_2\sigma_1)^2$, $\tau_3 = (\sigma_3\sigma_2\sigma_3)^2$, and $\Delta_4 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$.

Theorem 4.2.10 ([Jou25a])**.** *Let $n \in \mathbb{N}^*$ and $0 \leq m < n$ coprime with n. Then*

$$\mathrm{Stab}_{[m:n:m]}/\mathcal{BI}_4 = \begin{cases} \langle \tau_1\Delta_4,\ \sigma_2 \rangle & \text{if } n \geq 3, \\ \langle \tau_1\Delta_4,\ \sigma_2,\ \sigma_1\sigma_2^2\sigma_3 \rangle & \text{if } n = 2, \\ \langle \tau_1,\ \Delta_4,\ \sigma_2 \rangle & \text{if } n = 1. \end{cases}$$

Proof. For convenience, column vectors of $\mathcal{M}_{3,1}(\mathbb{Q})$ will be denoted in line : (u,v,w). Let $\beta \in \mathrm{Stab}_{[m:n:m]}$, and let $M = \rho_4^{int}(\beta)$.

$$M = \begin{pmatrix} a & e & b \\ x & f & y \\ c & g & d \end{pmatrix}.$$

Since the action of B_4 fixes $(1,0,1)$, one has

$$b = 1-a, \qquad d = 1-c, \qquad y = -x.$$

By definition, β stabilizes $[m:n:m]$, so either $M(m,n,m) = (m,n,m)$, or $M(m,n,m) = (-m,-n,-m)$.

• First, let us suppose that $M(m,n,m) = (m,n,m)$. Then the eigenspace associated to 1 has dimension more than 2, and $(0,1,0) = \frac{1}{n}(m,n,m) - m(1,0,1)$ belongs to this space. Therefore $e = g = 0$ and $f = 1$.
Moreover $\det(M) = 1$ implies that $c = a-1$. The matrix M is then

$$M = \begin{pmatrix} a & 0 & 1-a \\ x & 1 & -x \\ a-1 & 0 & 2-a \end{pmatrix}.$$

Multiplying β by σ_2^x to the left, we can suppose that $x = 0$. It is straightforward to check that for all $a \in \mathbb{Z}$,

$$\rho_4^{int}((\tau_1\Delta_4)^{a-1}) = \begin{pmatrix} a & 0 & 1-a \\ 0 & 1 & 0 \\ a-1 & 0 & 2-a \end{pmatrix},$$

so the matrix M is in the group generated by $\rho_4^{int}(\sigma_2)$ and $\rho_4^{int}(\tau_1\Delta_4)$.

• Now, let us suppose that $M(m,n,m) = (-m,-n,-m)$. This means

$$\begin{cases} ma + ne + m(1-a) = -m \\ nf = -n \\ mc + ng + m(1-c) = -m \end{cases}, \text{ so } e = g = -2\frac{m}{n}, \text{ and } f = -1.$$

Moreover, the matrix M has 1 and -1 as eigenvalues, and $\det(M) = 1$, therefore the eigenspace associated to -1 must have dimension 2. Taking the trace, we get $-1 = \mathrm{Tr}(M) = a + f + 1 - c$, so $c = a+1$. As before, we can multiply β by σ_2^{-x} to the left to get $x = 0$, so

$$M = \begin{pmatrix} a & -2m/n & 1-a \\ 0 & -1 & 0 \\ a+1 & -2m/n & -a \end{pmatrix}.$$

Notice that $M(0,1,0) = (-2m/n, -1, -2m/n)$, so $\rho_4^{int}(\beta)([0:1:0]) = [2m:n:2m]$. The points $[0:1:0]$ and $[2m:n:2m]$ are in the same orbit only if $n = 2$ or $n = 1$. So if $n \geq 3$, this case is impossible.

If $n = 2$, then $m = 1$ and

$$M = \begin{pmatrix} a & -1 & 1-a \\ 0 & -1 & 0 \\ a+1 & -1 & -a \end{pmatrix} = \rho_4^{int}((\tau_1\Delta_4)^{-a-1}\sigma_2(\sigma_1\sigma_2^2\sigma_3)).$$

Finally, if $n = 1$, then $m = 0$, and

$$M = \begin{pmatrix} a & 0 & 1-a \\ 0 & -1 & 0 \\ a+1 & 0 & -a \end{pmatrix} = \rho_4^{int}(\tau_1(\tau_1\Delta_4)^{a+1}),$$

and this completes the proof. □

Both in Corollay 4.2.5 and Theorem 4.2.10, the orbit of $[0:1:0]$ have a special role.

Notation 4.2.11. Denote by $\mathcal{O}_1 := \mathrm{Orb}_{B_4}([0:1:0])$ the *principal orbit* in $\mathbb{P}^2(\mathbb{Q})$.

4.2.2 Topology and geometry of orbits

In this paragraph, we investigate the way the orbits fill the rational projective plane. In particular, we highlight the symmetries of the orbits, that are carried by an affine property of the principal orbit $\mathcal{O}_1$. Finally, we outline an experimental dominance property of the principal orbit, that distinguishes it from the other orbits.

Note that the standard embedding into the real projective plane $\mathbb{P}^2(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{R})$ equips $\mathbb{P}^2(\mathbb{Q})$ with the natural topology induced from the Euclidean norm in $\mathbb{R}^3$. We will use this topology to study the question of density of orbits.

Proposition 4.2.12 ([Jou25a])**.** *Each orbit, except for the singleton orbit* $\{[1:0:1]\}$*, is dense in* $\mathbb{P}^2(\mathbb{Q})$.

Proof. Let us show that the orbit of $[m_0 : n_0 : m_0]$ is dense, for $n_0 \in \mathbb{N}^*$ and coprime m_0, n_0 such that $m_0 < n_0$. It suffices to check that for each representative $[m : n : m]$ with $m < n/2$, there exists a sequence of points in $\mathrm{Orb}_{B_4}([m_0 : n_0 : m_0])$ converging to $[m : n : m]$.

Consider the following sequence of points in $\mathbb{P}^2(\mathbb{Q})$

$$P_k := \left[m + \frac{n_0 + m_0 - m}{n_0 k + 1} : n + \frac{n_0 - n}{n_0 k + 1} : m + \frac{m_0 - m}{n_0 k + 1}\right] \underset{k \to +\infty}{\longrightarrow} [m : n : m].$$

For all $k \in \mathbb{N}^*$,

$$\begin{aligned} P_k &= [(n_0 k + 1)m + n_0 + m_0 - m : (n_0 k + 1)n + n_0 - n : (n_0 k + 1)m + m_0 - m] \\ &= [n_0(km + 1) + m_0 : n_0(kn + 1) : n_0 km + m_0], \end{aligned}$$

with $n_0(km + 1) + m_0$, $n_0(kn + 1)$, $n_0 km + m_0$ mutually prime because m_0 and n_0 are coprime. And

$$\gcd(n_0(km + 1) + m_0 - (n_0 km + m_0), n_0(kn + 1)) = \gcd(n_0, n_0(kn + 1)) = n_0,$$

therefore the point P_k is in the orbit of $[m_0 : n_0 : m_0]$, for all $k \in \mathbb{N}^*$. □

Proposition 4.2.12 implies that the orbits (except for the singleton orbit $\{[1:0:1]\}$) are neither closed nor open.

Proposition 4.2.13 ([Jou25a])**.** *The orbits are stable under the action of the dihedral group* D_4 *acting on the plane with origin placed at* $(1, 0)$.

Proof. The action of the dihedral group D_4 is generated by the rotation of $\pi/2$ around the point $(1, 0) = [1 : 0 : 1]$ and the symmetry with respect to the horizontal axis. Therefore, it is enough to check that for every point $[x : y : 1]$, the image by the rotation $[-y + 1 : x - 1 : 1]$ and the image by the symmetry $[x : -y : 1]$ are in the same orbit as $[x : y : 1]$.

Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$, be in reduced forms, and let $\delta = \gcd(b, d)$, with $b = \delta b'$, $d = \delta d'$ so

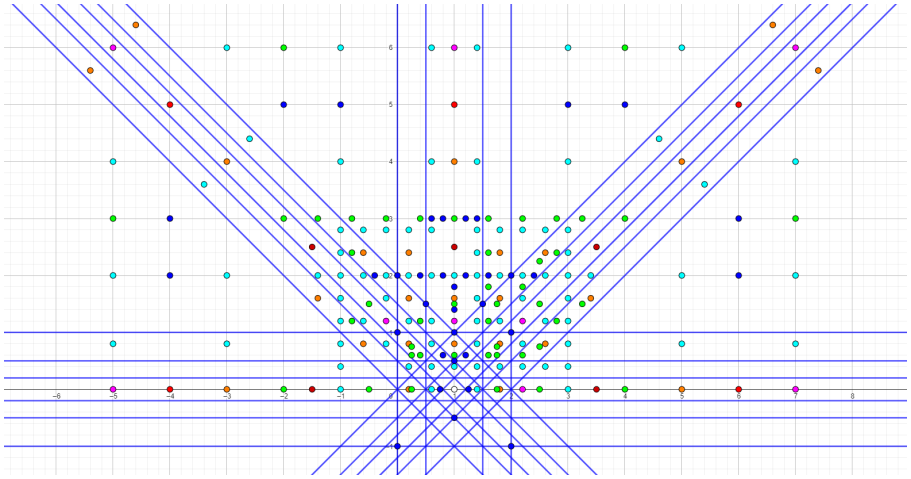


Figure 4.1: Sketch of the orbits in $\mathbb{P}^2(\mathbb{Q})$ with the affine chart $[x : y : 1]$. Each color represents an orbit, dark blue is for the principal orbit.

that $[x : y : 1] = [ad' : b'c : d'b]$ with ad', $b'c$ and $d'b$ are mutually prime. Then $[x : y : 1]$ is in the orbit of the representative $[m : n : m]$ with $n = \gcd(a-b, c)$ and $m \equiv ad' \mod (n)$.

Now $[x : -y : 1] = [ad' : -b'c : d'b]$ so it is also in the orbit of $[m : n : m]$.

Similarly,, $[-y+1 : x-1 : 1] = [-bc'+bd' : ad'-bd' : bd']$ with $\gcd(bc', ad'-bd') = n$ and $-bc'+bd' \equiv bd' \equiv m \mod (n)$, so the point is again in the same orbit. □

The principal orbit $\mathcal{O}_1$ has one particular property : among all orbits, it is the only one that contains infinitely many straight lines of $\mathbb{P}^2(\mathbb{Q})$, see Figure 4.1.

Proposition 4.2.14 ([Jou25a]). *Let $r/s \in \mathbb{Q}$.*
The affine line of $\mathbb{Q}^2$ (embedded in $\mathbb{P}^2(\mathbb{Q})$ by $(x, y) \mapsto [x : y : 1]$) having slope r/s and passing through the point $(0, c/d)$ is entirely in $\mathcal{O}_1$ if and only if

$$cs + rd = \pm \gcd(s, d).$$

Moreover, the only vertical lines that are in $\mathcal{O}_1$ are those of the form

$$D = \left\{ \left(\frac{r \pm 1}{r}, \lambda \right), \ \lambda \in \mathbb{Q} \right\} \quad \text{with } r \in \mathbb{N}^*.$$

Proof. Since c/d is in reduced form, the point $(0, c/d)$ is in $\mathcal{O}_1$.
Let us suppose that the line passing through $(0, c/d)$ and of slope r/s is in $\mathcal{O}_1$. Then for all $\lambda \in \mathbb{Q}$, the point $(\lambda, r/s\lambda + c/d)$ is in $\mathcal{O}_1$. We have

$$\left[\lambda : \frac{r}{s}\lambda + \frac{c}{d} : 1\right] = [\alpha ds : r\alpha d + c\beta s : \beta ds] = [\alpha ds_1 : r\alpha d_1 + c\beta s_1 : \beta ds_1],$$

where $d_1 = d/\gcd(s, d)$ and $s_1 = s/\gcd(s, d)$. As the three coordinates of this point are not necassarily mutually prime, the fact that the point is in $\mathcal{O}_1$ means that

$$\gcd(\alpha ds_1, \alpha rd_1 + \beta cs_1, ds_1\beta) = \gcd(ds_1(\alpha - \beta), \alpha rd_1 + \beta cs_1).$$

or, because α and β are coprime,

$$\gcd(ds_1, \alpha r d_1 + \beta c s_1) = \gcd(ds_1(\alpha - \beta), \alpha r d_1 + \beta c s_1).$$

In particular, if we take $\alpha = \beta = 1$, then we get that $rd_1 + cs_1$ divides ds_1.
Note that it is means that $\gcd(r, c)$ divides d or s, so actually $\gcd(r, c) = 1$.

Now take $\alpha = (c + d)s_1$ and $\beta = (s - r)d_1$, so that $\alpha r d_1 + \beta c s_1 = (rd_1 + cs_1)ds_1$. We get

$$ds_1 = \pm(rd_1 + cs_1)ds_1,$$

so $rd_1 + cs_1 = \pm 1$.
For this argument to be valid, we need to check that the α and β chosen above are coprime. We check that β is coprime to $\alpha - \beta$, which is equivalent.

$$\begin{aligned}
\gcd(\alpha - \beta, \beta) &= \gcd(rd_1 + cs_1, (s - r)d_1) \\
&= \gcd(rd_1 + cs_1, s - r) \text{ because } \gcd(d_1, cs_1) = 1 \\
&= \gcd(rd_1 + cs_1, r) \text{ because } rd_1 + cs_1 \text{ divides } s \\
&= \gcd(cs_1, r) \\
&= 1,
\end{aligned}$$

this concludes the argument.

Conversely, let us suppose that $rd_1 + cs_1 = \pm 1$. Let $\lambda = \alpha/\beta$ in reduced form.
Let us check the criteria for the point $[\alpha ds : r\alpha d + c\beta s : \beta ds] = [\alpha ds_1 : r\alpha d_1 + c\beta s_1 : \beta ds_1]$ to be in $\mathcal{O}_1$. First note that the greatest common divisor of αds_1, βds_1 and $r\alpha d_1 + c\beta s_1$ is $\gcd(ds_1, r\alpha d_1 + c\beta s_1)$. One has

$$\begin{aligned}
\gcd(\alpha ds_1 - \beta ds_1, r\alpha d_1 + c\beta s_1) &= \gcd((\alpha - \beta)ds_1, rd_1(\alpha - \beta) \pm \beta) \quad \text{because } rd_1 + cs_1 = \pm 1 \\
&= \gcd(ds_1, rd_1(\alpha - \beta) \pm \beta) \\
&= \gcd(ds_1, r\alpha d_1 + c\beta s_1).
\end{aligned}$$

Therefore, when divided by $\gcd(ds_1, r\alpha d_1 + c\beta s_1)$, we get that $[\alpha ds_1 : r\alpha d_1 + c\beta s_1 : \beta ds_1]$ is in $\mathcal{O}_1$, and so the entire line is in the principal orbit.

For the vertical lines, let us suppose that the line $\{(a/b, \lambda)\}$ is in $\mathcal{O}_1$. Then the point $[a/b : 0 : 1] = [a : 0 : b]$ is in $\mathcal{O}_1$, so $\gcd(a - b, 0) = 1$, meaning that $a - b = \pm 1$.

Conversely, it is straightforward to check that the line $\{((r \pm 1)/r, \lambda)\}$ is entierly in $\mathcal{O}_1$. Proposition 4.2.14 is proved. □

Let us explain why in a certain sense, the principal orbit is much bigger than the others. Although we have no precise statement, computer experimentations clearly demonstrate this phenomenon.

Definition 4.2.15. Let A be a subset of $\mathbb{Q}^2_{>0}$. Let $d \in \mathbb{N}^*$. Let S_d be the set of positive rational numbers with numerator and denominator lower that d (when written in reduced form). We say that A has *rational density* α when :

$$\frac{\sharp(A \cap S_d^2)}{\sharp S_d^2} \underset{d \to +\infty}{\longrightarrow} \alpha.$$

Conjecture 4.2.16. *The principal orbit $\mathcal{O}_1$ has a rational density greater than* 0.75.

Remark 4.2.17. As the natural density for positive integers, this rational density is a way to state the visual intuition that the principal orbit takes the major part of the projective plane (around 3/4 of the plane). This definition of rational density was inspired by the work done in [LM22].

The following conjecture relies on the exact computation of the values $\sharp(\mathcal{O}_1 \cap S_d^2)/\sharp S_d^2$ for large d (until $d = 800$), see Figure 4.2.

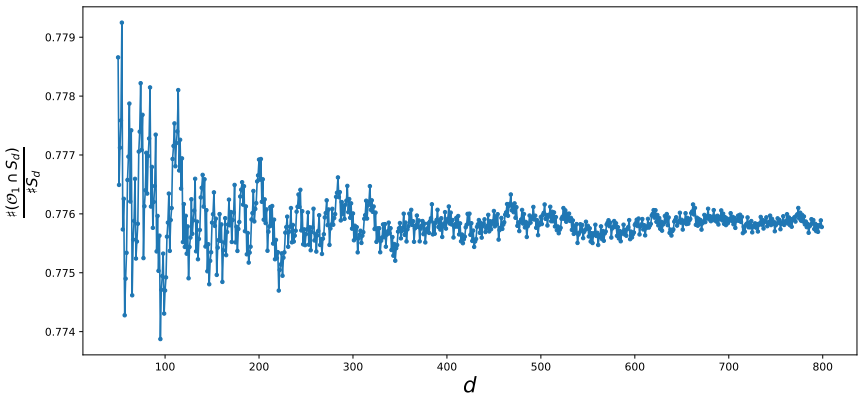


Figure 4.2: Partial rational density of $\mathcal{O}_1$ for $d = 50, \ldots, 800$.

4.3 Deformation of the rational plane

Now that the case $q = 1$ is well understood, we can come back to the quantum setting and let the braid group B_4 act on the rational plane to deform it thanks to the Burau representation. The study of the case $q = 1$ motivates us to focus on the principal orbit.

4.3.1 Deformation of the principal orbit

Recall the Burau representation of the braid group B_4.

$$\rho_4 : B_4 \to \mathrm{GL}_3(\mathbb{Z}[q, q^{-1}]),$$

$$\rho_4(\sigma_1) = \begin{pmatrix} q & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_4(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ -q & q & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_4(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & q \end{pmatrix}.$$

This representation defines an action $B_4 \curvearrowright \mathbb{P}^2(\mathbb{Z}(q))$ by

$$\begin{aligned} \rho_4(\sigma_1) : \quad [r : s : t] &\mapsto [qr + s : s : t], \\ \rho_4(\sigma_2) : \quad [r : s : t] &\mapsto [r : qs + t - qr : t], \\ \rho_4(\sigma_3) : \quad [r : s : t] &\mapsto [r : s : qt - qs]. \end{aligned} \tag{4.6}$$

Definition 4.3.1. [Jou25a] By analogy with the case where $q = 1$, let us denote by $\mathcal{O}_q$ the orbit of the point $[0 : 1 : 0] \in \mathbb{P}^2(\mathbb{Z}(q))$, under the action of B_4 via the reduced Burau representation ρ_4, and let us call it the *q-deformed principal orbit*.

Remark 4.3.2. With this definition, a point $[r:s:t] \in \mathcal{O}_1$ has infinitely many quantizations, depending on the braid chosen to connect $[r:s:t]$ to $[0:1:0]$.

Given a braid β such that $\rho_4^{int}(\beta)([0:1:0]) = [r:s:t]$, the quantizations of the point $[r:s:t]$ are reached by $\rho_4(\beta\,\mathrm{Stab}_{[0:1:0]})([0:1:0])$. Yet, the generators computed in Theorem 4.2.10 have a trivial action on $[0:1:0]$, as

$$\rho_4(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ -q & q & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_4(\tau_1) = \begin{pmatrix} -q^3 & 0 & q+1 \\ 0 & -q^3 & -q^2+1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_4(\Delta_4) = \begin{pmatrix} 0 & 0 & q \\ 0 & -q^2 & 0 \\ q^3 & 0 & 0 \end{pmatrix}.$$

Therefore, the split of the point $[r:s:t]$ in several quantizations actually comes from the braid Torelli group $\mathcal{BI}_4$. For instance the q-action of the braid $\sigma_3\tau_1^2\sigma_3^{-1}$ is given by

$$\rho_4(\sigma_3\tau_1^2\sigma_3^{-1}) = \begin{pmatrix} q^6 & q^5+q^4-q^2-q & -q^5-q^4+q^2+q \\ 0 & q^4+q^3-q & q^6-q^4-q^3+q \\ 0 & q^4+q^3-q-1 & q^6-q^4-q^3+q+1 \end{pmatrix}.$$

Definition 4.3.3. The quantization map is a set-valued function defined by

$$\begin{array}{rccc} \mathcal{Q}: & \mathcal{O}_1 & \longrightarrow & \mathcal{P}(\mathbb{P}^2(\mathbb{Z}(q))) \\ & p & \longmapsto & \{\rho_4(\beta)([0:1:0]) \mid \beta \text{ s.t. } \rho_4^{int}(\beta)([0:1:0]) = p\} \end{array}.$$

For $p \in \mathcal{O}_1$, the image $\mathcal{Q}(p)$ is called the *quantization* of p, and an element $[\mathcal{R}:\mathcal{S}:\mathcal{T}]$ of $\mathcal{Q}(p)$ will be called a *deformation* of p.

Remark 4.3.4. According to the previous remark, for $p \in \mathcal{O}_1$, if we denote by β_p the braid provided by the braided Euclidean algorithm such that $\rho_4^{int}(\beta_p)([0:1:0]) = p$, we have

$$\mathcal{Q}(p) = \{\rho_4(\beta_p\gamma)([0:1:0]) \mid \gamma \in \mathcal{BI}_4\}.$$

Example 4.3.5. Let us quantize the point $[7:18:14]$.

(i) Thanks to the braided Euclidean algorithm, we compute the braid $\beta = \sigma_2^2\sigma_1\sigma_3^{-3}\sigma_2^{-3}\sigma_1^3\sigma_3^{-2}$ such that $\rho_4^{int}(\beta)([0:1:0]) = [7:18:14]$. Then

$$\rho_4(\beta)([0:1:0]) = [\mathcal{R}(q):\mathcal{S}(q):\mathcal{T}(q)], \text{ with}$$

$$\begin{aligned} \mathcal{R}(q) &= q^9+2q^8+3q^7+3q^6+q^5-q^4-q^3-q^2\ , \\ \mathcal{S}(q) &= -q^{11}-2q^{10}-2q^9+q^8+6q^7+11q^6+10q^5+4q^4-q^3-4q^2-3q-1\ , \\ \mathcal{T}(q) &= q^8+3q^7+6q^6+6q^5+3q^4-2q^2-2q-1. \end{aligned}$$

Note that these polynomials have both positive and negative coefficients, and the positive (resp. the negative) parts are unimodal.

(ii) Let us also compute another point in the quantization of $[7:18:14]$, with $\rho_4(\beta\sigma_3\tau_1^2\sigma_3^{-1})([0:1:0]) = [\tilde{\mathcal{R}}(q):\tilde{\mathcal{S}}(q):\tilde{\mathcal{T}}(q)]$,

$$\tilde{\mathcal{R}}(q) = q^{15}+2q^{14}+3q^{13}+3q^{12}-3q^{10}-3q^9+3q^7+3q^6+q^5-q^4-q^3-q^2\ ,$$

$\tilde{\mathcal{S}}(q) = -q^{17} - 2q^{16} - 2q^{15} + q^{14} + 7q^{13} + 13q^{12} + 11q^{11} + q^{10} - 9q^9 - 10q^8 - 2q^7 + 7q^6 + 9q^5 + 4q^4 - q^3 - 4q^2 - 3q - 1$,
$\tilde{\mathcal{T}}(q) = q^{14} + 3q^{13} + 6q^{12} + 6q^{11} + 2q^{10} - 3q^9 - 5q^8 - 2q^7 + 3q^6 + 5q^5 + 3q^4 - 2q^2 - 2q - 1$.
Again, these polynomials have positive and negative parts, and each part is unimodal.

We prove that the quantization of the principal orbit is compatible with the q-deformation of the projective rational line defined by Morier-Genoud and Ovsienko (see Section 1.2).

Consider the following embedding of the projective line into the projective plane

$$\mathbb{P}^1(\mathbb{Q}) \overset{\iota}{\hookrightarrow} \mathbb{P}^2(\mathbb{Q}), \qquad [r:s] \mapsto [r:s:0].$$

It is easy to see that its image belongs to the principal orbit $\mathcal{O}_1$. Similarly, we define the embedding

$$\mathbb{P}^1(\mathbb{Z}(q)) \overset{\iota_q}{\hookrightarrow} \mathbb{P}^2(\mathbb{Z}(q)), \qquad [\mathcal{R}(q):\mathcal{S}(q)] \mapsto [\mathcal{R}(q):\mathcal{S}(q):0].$$

We will prove that the above embeddings commute with quantization.

Theorem 4.3.6 ([Jou25a])**.** *The quantization of the projective line in the sense of Morier-Genoud and Ovsienko and our quantization commute with the embedding. In other words, for every $[r:s] \in \mathbb{P}^1(\mathbb{Q})$, we have*

$$\iota_q(\mathcal{Q}([r:s])) \in \mathcal{Q}(\iota([r:s])).$$

The other natural embedding of the projective line into the projective plane, namely $[r:s] \mapsto [0:r:s]$ also commutes with our quantization map.

Proof. Let r and s be two coprime integers, with $s > 0$, and such that

$$\mathcal{Q}([r:s]) = [\mathcal{R}(q):\mathcal{S}(q)]_q,$$

with $\mathcal{R}(q)$, $\mathcal{S}(q)$ two coprime polynomials.

Let us apply the braided Euclidean algorithm to the point $[r:s:0]$.
As the third coordinate of the point is zero, the algorithm follows exactly the steps of the usual Euclidean algorithm, so the braid provided is

$$\beta = \sigma_1^{-a_{2k+1}}\sigma_2^{a_{2k}} \cdots \sigma_1^{-a_3}\sigma_2^{a_2}\sigma_1^{-a_1},$$

with the a_i's being the coefficients of the odd positive continued fraction (see Notation 1.1.3) :

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{2k+1}}}} = [a_1, a_2, \cdots, a_{2k+1}].$$

One of the quantizations of $[r:s:0]$ is $\rho_4(\beta^{-1}) \cdot [0:1:0]$.

$$\rho_4(\beta^{-1}) = \rho_4(\sigma_1)^{a_1}\rho_4(\sigma_2)^{-a_2} \cdots \rho_4(\sigma_1)^{a_{2k+1}} = \begin{pmatrix} M_q & \star \\ 0 & 1 \end{pmatrix},$$

where M_q is the image of the 3-strands braid β by the Burau representation of B_3, because of the following commutative diagramme.

$$\begin{array}{ccc} & \overset{\rho_3}{\longrightarrow} & \\ B_3 & & \mathrm{GL}_2(\mathbb{Z}[q,q^{-1}]) \\ \sigma_i \mapsto \sigma_i \big\downarrow & & \big\downarrow M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \\ B_4 & \underset{\rho_4}{\longrightarrow} & \mathrm{GL}_3(\mathbb{Z}[q,q^{-1}]) \end{array}$$

As a 3-strands braid, β sends $[0:1]$ on $[r:s]$, so the second column of M_q is the unique deformation of $[r:s]$ in the sense of [MGO20], that is $(\mathcal{R}(q), \mathcal{S}(q))$, and thus $[\mathcal{R}(q):\mathcal{S}(q):0]$ is among the quantizations of the point $[r:s:0]$.

Note that the matrix M_q appearing in top-left block of $\rho_4(\beta^{-1})$ is closely related to the q-matrix $M_{+,q}\left(\frac{r}{s}\right)$, defined in Chapter 1 (Definition 1.20) : $M_q = M_{+,q}\left(\frac{r}{s}\right) L_q^{-1} T_q$. □

4.3.2 Particular specializations

The obstruction to the unicity of quantization comes from the fact that the inclusion $\ker(\rho_4) \subset \mathcal{BI}_4$ is strict (the braid Torelli group is "too big"). We investigate specializations of q at complex values for which the reverse inclusion holds. Recall the evaluation map defined in Definition 4.1.13.

Notation 4.3.7. Let $j := \mathrm{e}^{\frac{2i\pi}{3}}$ be one of the third roots of unity.

Lemma 4.3.8. *The only complex values z for which $\mathcal{BI}_4 \subset \ker(\mathrm{ev}_z \rho_4)$ are $1, -1, j, j^2$.*

Proof. Recall that $\mathcal{BI}_4$ is normally generated by τ_1^2, τ_3^2 and Δ_4^2 (see Notation 4.2.9), and that $\rho_4(\Delta_4^2) = q^4 \,\mathrm{Id}$.
We can compute for all $k \in \mathbb{N}$,

$$\rho_4(\tau_1^2)^k = \begin{pmatrix} q^{6k} & 0 & [k]_{q^6} f_1(q) \\ 0 & q^{6k} & [k]_{q^6} f_2(q) \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{with } \begin{cases} f_1(q) = -q^4 - q^3 + q + 1 = -(q-1)(q+1)(q^2+q+1) \\ f_2(q) = q^5 - q^3 - q^2 + 1 = (q-1)^2(q+1)(q^2+q+1) \end{cases}.$$

Likewise,

$$\rho_4(\tau_3^2)^k = \begin{pmatrix} 1 & 0 & 0 \\ -[k]_{q^6} g_1(q) & q^{6k} & 0 \\ -[k]_{q^6} g_2(q) & 0 & q^{6k} \end{pmatrix},$$

$$\text{with } \begin{cases} g_1(q) = q^6 - q^4 - q^3 + q = q(q-1)^2(q+1)(q^2+q+1) \\ g_2(q) = q^6 + q^5 - q^3 - q^2 = -q^2(q-1)(q+1)(q^2+q+1) \end{cases}.$$

If $q = z \in \{1, -1, j, j^2\}$, then $\mathrm{ev}_z(\rho_4(\tau_1^2)) = \mathrm{ev}_z(\rho_z(\tau_3^2)) = \mathrm{Id}$, so the whole normal group generated by τ_1^2, τ_3^2 and Δ_4^2 is in $\ker(\mathrm{ev}_z \rho_4)$.

Conversely, if $z \notin \{1, -1, j, j^2\}$, then $f_1(z) \neq 0$ so $\mathrm{ev}_z(\rho_4(\tau_1^2)) \neq \mathrm{Id}$, and $\mathcal{BI}_4 \not\subset \ker(\mathrm{ev}_z \rho_4)$. □

Remark 4.3.9. For $q = 1$, the statement is obvious, as $\ker(\mathrm{ev}_1 \rho_4) = \ker(\rho_4^{int}) = \mathcal{BI}_4$ by definition. For $q = -1$, we have $\ker(\mathrm{ev}_{-1} \rho_4) = P_4$, the pure braid group with 4 strands. For $q = z$ a primitive third root of unity, the inclusion $\mathcal{BI}_4 \subset \ker(\mathrm{ev}_z \rho_4)$ is strict. For instance, $\mathrm{ev}_z(\rho_4(\sigma_1^3)) = \mathrm{Id}$.

Definition 4.3.10. Let $p = [r : s : t] \in \mathcal{O}_1$. The j-deformed point associated to p is defined by

$$[r : s : t]_j := \mathrm{ev}_j(\rho_4(\beta))([0 : 1 : 0]),$$

for any $\beta \in B_4$ such that $\rho_4^{int}(\beta)([0 : 1 : 0]) = p$.

Example 4.3.11. Let $p = [1 : 5 : 3]$. The braid $\beta = \sigma_2^2\sigma_1\sigma_3^{-3}$ sends $[0 : 1 : 0]$ to p. Then the j-analogue of p is

$$[1 : 5 : 3]_j = [1 : -j : 0].$$

Remark 4.3.12. During experimentations, we noticed that for every point $[r : s : t] \in \mathcal{O}_1$ we looked at, the j-analogue $[R : S : T] = [r : s : t]_j$ satisfies that R and T are either 0 or invertible in $\mathbb{Z}[j]$ (that is $\mathcal{N}(R) \leq 1$, $\mathcal{N}(T) \leq 1$) and that $\mathcal{N}(S) \in \{0, 1, 3\}$, where $\mathcal{N}$ denote the norm of the ring of integers $\mathbb{Z}[j]$.

4.3.3 Minimal unimodal quantization

The aim of this paragraph is to choose one particular deformation of a point $p \in \mathcal{O}_1$ among the quantization $\mathcal{Q}(p)$.

Definition 4.3.13. Let $\mathcal{R}(q) \in \mathbb{Z}[q]$. We say that $\mathcal{R}$ is *piecewise unimodal* when its sequence of coefficients $(a_0, a_1, \cdots, a_n)$ is divided in subsequences of alternatively positive and negative coefficients $(+|a_0|, ..., +|a_{i_1}|)$, $(-|a_{i_1+1}|, \cdots, -|a_{i_2}|)$, ..., each subsequence being unimodal.
Let $[\mathcal{R} : \mathcal{S} : \mathcal{T}] \in \mathbb{P}^2(\bar{\Lambda})$, renormalized such that $\mathcal{R}, \mathcal{S}$ and $\mathcal{T}$ are polynomials in q. We say that $[\mathcal{R} : \mathcal{S} : \mathcal{T}]$ is *fully piecewise unimodal* when $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ are piecewise unimodal.

Example 4.3.14. The deformation of $[3 : 6 : 4]$ obtained via the braided Euclidean algorithm is

$$[q^5 + q^4 + q^3, -q^{10} - 2q^9 - 2q^8 - 2q^7 - q^6 + q^5 + 3q^4 + 4q^3 + 3q^2 + 2q + 1, q^3 + q^2 + q + 1].$$

This deformation is fully piecewise unimodal. In particular, in the second coordinate the sequence of coefficients is $(1, 2, 3, 4, 3, 1, -1, -2, -2, -2, -1)$, so it has two pieces and each is unimodal.

Remark 4.3.15. Some deformations of points in the principal orbit are not fully piecewise unimodal. For instance, in $\mathcal{Q}([21 : 29 : 11])$, there is $[\mathcal{R}(q) : \mathcal{S}(q) : \mathcal{T}(q)]$ with

$$\mathcal{S}(q) = q^{13} + 3q^{12} + 6q^{11} + 8q^{10} + 8q^9 + 5q^8 + 2q^7 + q^6 + 2q^5 + 2q^4 - q^3 - 3q^2 - 3q - 2,$$

which is not piecewise unimodal.

Conjecture 4.3.16. *Let $p \in \mathcal{O}_1$. There is a unique deformation $[\mathcal{R} : \mathcal{S} : \mathcal{T}]$ of p in $\mathcal{Q}(p)$ such that $\deg(\mathcal{R})$, $\deg(\mathcal{S})$ and $\deg(\mathcal{T})$ are minimal.*

Definition 4.3.17. Assuming the conjecture above, we can define the deformation of a point $p \in \mathcal{O}_1$ to be the minimal (in degrees) deformation of p.
If $p = [r : s : t]$, we denote this minimal deformation by $[r : s : t]_q$.

Remark 4.3.18. This definition would match with the deformation of the projective rational line embedded in $\mathbb{P}^2(\mathbb{Q})$, because if $[r:s]_q = [\mathcal{R}:\mathcal{S}]$, then $[\mathcal{R}:\mathcal{S}:0]$ is minimal by unicity of the deformation of $[r:s]$. Moreover in this case, the deformation of $[r:s:0]$ is fully piecewise unimodal.

To support our conjecture, we sum up some of the examples we computed.

Example 4.3.19 (Non unimodality)**.** Using the braided Euclidean algorithm, we computed deformations of $[r:s:t]$ for all the triples of nonnegative integers (r,s,t) (satisfying the condition $\gcd(r-t,s)=1$ to be in $\mathcal{O}_1$) bounded by 100. In this set of examples, only 1518 (over 302172) were not fully piecewise unimodular, the first one (for the lexicographic order) occuring for the triplet $(10,67,3)$, for which the polynomials are

$\mathcal{R}(q) = -q^{15} - 2q^{14} - q^{13} + q^{12} + 4q^{11} + 5q^{10} + 3q^9 + q^8,$
$\mathcal{S}(q) = -q^{15} - 3q^{14} - 4q^{13} - 3q^{12} + q^{11} + 6q^{10} + 8q^9 + 8q^8 + 7q^7 + 8q^6 + 9q^5 + 10q^4 + 9q^3 + 7q^2 + 4q + 1,$
$\mathcal{T}(q) = q^{10} + q^9 + q^8.$

For every point whose deformation using the braided Euclidean algorithm was not fully piecewise unimodal, we found an other deformation that is fully piecewise unimodal, by applying variations of the braided Euclidean algorithm. For instance, the braid $\beta = \sigma_2^{-10}\sigma_1^{-3}\sigma_3\sigma_2^2\sigma_1^{-1}$ satisfies $\rho_4^{int}(\beta)([0:1:0]) = [10:67:3]$, and we have $\rho_4(\beta)([0:1:0]) = [\tilde{\mathcal{R}}(q):\tilde{\mathcal{S}}(q):\tilde{\mathcal{T}}(q)]$ with

$\tilde{\mathcal{R}}(q) = -q^{14} - 2q^{13} - 3q^{12} - 2q^{11} - q^{10} - q^9,$
$\tilde{\mathcal{S}}(q) = q^{15} + q^{14} - 3q^{12} - 5q^{11} - 6q^{10} - 7q^9 - 7q^8 - 7q^7 - 7q^6 - 7q^5 - 7q^4 - 6q^3 - 4q^2 - 2q - 1,$
$\tilde{\mathcal{T}}(q) = -q^{16} - q^{15} - q^{14}.$

which are piecewise unimodal.
However, we can find a third deformation of $[10,67,3]$ such that the degrees of the three coordinates are together minimal. Indeed with $\beta'' = \sigma_2^{-9}\sigma_1^2\sigma_2^3\sigma_1^2\sigma_3^{-3}\sigma_2\sigma_3\tau_1^2\tau_3(\sigma_2\sigma_3)^2$, we get $\rho_4(\beta'') = [\mathcal{R}'':\mathcal{S}'':\mathcal{T}'']$,

$\mathcal{R}''(q) = q^{12} + 2q^{11} + 3q^{10} + 3q^9 + q^8,$
$\mathcal{S}''(q) = q^{12} + 3q^{11} + 6q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 7q^5 + 7q^4 + 6q^3 + 4q^2 + 2q + 1,$
$\mathcal{T}''(q) = q^{10} + q^9 + q^8.$

Example 4.3.20 (Minimality of degrees)**.** Let us focus on three significative examples, the points $[2:1:1]$, $[3:1:5]$ and $[21:29:11]$. For these three points, we computed many different deformations in their quantizations.
Indeed, for a given point p, one can compute the braid given by the braided Euclidean algorithm, β_p, and then multiply by any element of $\mathcal{BI}_4$ to get an other deformation of the point.

Let us denote by $\sigma(a,b,c)$ the braid $\sigma_1^a\sigma_2^b\sigma_3^c$ with $a,b,c \in \mathbb{Z}$.
For each point p, we computed the deformations corresponding to the braids $\beta_p\gamma\tau\gamma^{-1}$, for $\tau \in \{\tau_1^2, \tau_1^2\tau_3^2\}$ and $\gamma = \sigma(a_1,b_1,c_1)\sigma(a_2,b_2,c_2)\dots\sigma(a_N,b_N,c_N)$, with $-4 \le a_i, b_i, c_i < 4$

and $0 < N < 3$. These parameters were chosen according to the computation power of our computer. With this method, we reach 16514 braids.

• Example 1 : $p = [2 : 1 : 1]$. The braided Euclidean algorithm gives $\beta_p = \sigma_1^2\sigma_3^{-1}$, and the corresponding deformation is $[q+1 : 1 : 1]$.
Among the 16514 deformations we looked at, the deformation $[q+1 : 1 : 1]$ is minimal in degrees. There was 3522 non fully piecewise unimodal deformations, the minimal one (in degrees) being

$$\begin{aligned}\mathcal{R}(q) = &- q^{20} - 4q^{19} - 8q^{18} - 10q^{17} - 7q^{16} + 7q^{14} + 10q^{13} + 11q^{12} + 12q^{11} + 11q^{10} + 5q^9 \\ &- 3q^8 - 7q^7 - 6q^6 - 2q^5 - q^4 - 2q^3 - 2q^2 - q,\end{aligned}$$

$$\begin{aligned}\mathcal{S}(q) = &-q^{19} - 3q^{18} - 5q^{17} - 5q^{16} - 2q^{15} + 2q^{14} + 4q^{13} + 5q^{12} + 7q^{11} + 9q^{10} + 7q^9 - 5q^7 \\ &- 6q^6 - 2q^5 - q^3 - 2q^2 - q,\end{aligned}$$

$$\begin{aligned}\mathcal{T}(q) = &- q^{19} - 4q^{18} - 8q^{17} - 10q^{16} - 7q^{15} + 6q^{13} + 9q^{12} + 11q^{11} + 13q^{10} + 12q^9 + 5q^8 \\ &- 3q^7 - 8q^6 - 6q^5 - 2q^4 - q^3 - 2q^2 - 2q - 1.\end{aligned}$$

• Example 2 : $p = [3 : 1 : 5]$. We applied the same protocole to this second point. Here the braided Euclidean algorithm returns $\beta_p = \sigma_1^3\sigma_3^{-5}$, leading to the deformation $[q^6+q^5+q^4 : q^4 : q^4+q^3+q^2+q+1]$.
There was 2737 non fully piecewise unimodal deformations.
However, we found that the braid $\beta' = \sigma_1^3\sigma_3^{-5}\sigma_2\sigma_3\tau_1^2\tau_3^2(\sigma_2\sigma_3)^2$ provides polynomials of lower degrees than with β_p, indeed

$$\rho_4(\beta')([0 : 1 : 0]) = [q^4 + 2q^3 + q^2 - 1 : q^3 + q^2 - 1 : q^3 + 2q^2 + 2q].$$

Therefore even if the braided Euclidean algorithm is efficient, it is not always the most efficient. It seams that $[q^4 + 2q^3 + q^2 - 1 : q^3 + q^2 - 1 : q^3 + 2q^2 + 2q]$ is the minimal deformation for $[3 : 1 : 5]$.

• Example 3 : $p = [21 : 29 : 11]$. In this example, the braided Euclidean algorithm leads to a non fully piecewise unimodal deformation of p. In this example, there are 2219 non fully piecewise unimodal deformations.
The minimal deformation seams to be the one given by the braided Euclidean algorithm. The lowest deformation we found that is fully piecewise unimodal is

$$\mathcal{R}(q) = q^{14} + 2q^{13} + 4q^{12} + 5q^{11} + 5q^{10} + 3q^9 + q^8 + q^6 + 2q^5 + q^4 - q^3 - 2q^2 - q$$
$$\mathcal{S}(q) = q^{14} + 3q^{13} + 6q^{12} + 8q^{11} + 8q^{10} + 5q^9 + q^8 - q^7 + q^6 + 4q^5 + 3q^4 - q^3 - 4q^2 - 4q - 1$$
$$\mathcal{T}(q) = q^{12} + 2q^{11} + 3q^{10} + 3q^9 + 2q^8 - q^6 + q^4 + q^3 - q.$$

4.4 Higher braid groups B_n

To pave the way for future works, we study the action of B_{n+2} on $\mathbb{P}^n(\mathbb{Q})$ for all $n \geq 1$, defined via the integral Burau representation $\rho_n^{int} := \mathrm{ev}_1 \circ \rho_n$. On the standard generators,

this action is

$$\sigma_i \cdot [x_1 : \cdots : x_{i-1} : x_i : x_{i+1} \cdots : x_{n+1}] = [x_1 : \cdots : x_{i-1} : x_i + x_{i+1} - x_{i-1} : x_{i+1} : \cdots : x_{n+1}] \tag{4.7}$$

Notation 4.4.1. As for the cases $n = 1$ and $n = 2$, a point in $\mathbb{P}^n(\mathbb{Q})$ will be represented by a $(n+1)$-uple with mutually coprime integer coordinates $[x_1 : x_2 : \cdots : x_{n+1}]$. There are two such representatives for a given point that differ by the sign. With this notation, the standard generator σ_i only changes the coordinate x_i.

We will study the action $B_{n+2} \curvearrowright \mathbb{P}^n(\mathbb{Q})$ inductively, by exploiting natural inclusions $B_m \subset B_{n+2}$, for $m < n$.
More precisely, let $k \in [\![1, n+1]\!]$, and define the natural inclusions

$$\iota^\ell : \begin{array}{ccc} B_k & \longrightarrow & B_{n+2} \\ \sigma_i & \longmapsto & \sigma_i \end{array} \quad \text{and} \quad \iota^r : \begin{array}{ccc} B_k & \longrightarrow & B_{n+2} \\ \sigma_i & \longmapsto & \sigma_{n-k+i} \end{array}.$$

We say that $\iota^\ell(B_k)$ is the *left copy* of B_k inside B_{n+2}, and that $\iota^r(B_k)$ is the right copy of B_k inside B_{n+2}.
From this, consider the inclusion $B_k \times B_{n-k+2} \subset B_{n+2}$:

$$\iota_{k,n+2} : \begin{array}{ccc} B_k \times B_{n-k+2} & \longrightarrow & B_{n+2} \\ (\beta, \gamma) & \longmapsto & \iota^\ell(\beta)\iota^r(\gamma) \end{array}$$

Then $B_k \times B_{n-k+2}$ naturally acts on the set of points whose kth coordinate is zero: $p = [x_1 : \cdots : x_{k-1} : 0 : x_{k+1} : \cdots : x_{n+1}]$. The left copy of B_k acts on the $k-1$ first coordinates and the right copy of B_{n-k+1} on the last $n-k+1$ coordinates. This action is compatible with the action we have defined in 4.5.

Our main ingredient will be the next technical lemma. Its proof is postponed to the end of the section.

Lemma 4.4.2. *Let $n \in \mathbb{N}^*$. Let $p \in \mathbb{P}^n(\mathbb{Q})$. Then there is a braid $\beta \in B_{n+2}$ such that $\beta \cdot p$ is of the form $[x_1 : 0 : x_3 : x_4 : \cdots : x_{n+1}]$.*

4.4.1 Case of odd n

Theorem 4.4.3. *Let $n = 2m+1$ be an odd positive integer. Then the action $B_{n+2} \curvearrowright \mathbb{P}^n(\mathbb{Q})$ as in (4.7) has $m+1$ orbits, and a system of representatives is given by*

$$\{[1 : 0 : 0 : \varepsilon_1 : 0 : \varepsilon_2 : 0 : \cdots : \varepsilon_m] \mid \varepsilon_i \in \{0, 1\} \text{ and } \varepsilon_i \leq \varepsilon_{i+1}\}.$$

The proof of Theorem 4.4.3 is written in the end of the subsection. It relies on some intermediaries results. We begin by identifying an invariant of the action $B_{n+2} \curvearrowright \mathbb{P}^n(\mathbb{Q})$.

Definition 4.4.4. Let $[x_1 : x_2 : \cdots : x_{n+1}]$ be a point of $\mathbb{P}^n(\mathbb{Q})$, with the x_i's mutually prime. An *odd component* of this point is a maximal sequence of successive odd coordinates.

Example 4.4.5. The point $[3 : 2 : 1 : -3 : 5]$ has two odd components, (3) and $(1, -3, 5)$.

Proposition 4.4.6. *The number of odd components is invariant under the action $B_{n+2} \curvearrowright \mathbb{P}^n(\mathbb{Q})$, for n odd.*

Proof. Let $p = [x_1 : x_2 : \cdots : x_{n+1}]$ be any point of $\mathbb{P}^n(\mathbb{Q})$. Recall that we choose coordinates x_i's to be mutually prime integers. Reduce the coordinates modulo 2 : even coordinates are replaced by 0 and odd coordinates by 1. The action of B_{n+2} commutes with the reduction modulo 2. For $k \in [\![2, n]\!]$, the action of σ_k modulo 2 only changes the coordinate x_k, according to the values of x_{k-1} and x_{k+1}. There are eight cases :

$$\begin{array}{ccc}
(x_{k-1}, x_k, x_{k+1}) & \longmapsto & (x_{k-1}, x_k + x_{k+1} - x_{k-1}, x_{k+1}) \\
(0,0,0) & \longmapsto & (0,0,0) \\
(0,0,1) & \longmapsto & (0,1,1) \\
(0,1,0) & \longmapsto & (0,1,0) \\
(0,1,1) & \longmapsto & (0,0,1) \\
(1,0,0) & \longmapsto & (1,1,0) \\
(1,0,1) & \longmapsto & (1,0,1) \\
(1,1,0) & \longmapsto & (1,0,0) \\
(1,1,1) & \longmapsto & (1,1,1)
\end{array} .$$

In any case, no new connected component of 1 is created, and no one is deleted, therefore the number of such components is invariant by σ_k. The same phenomenon can be checked for the action of σ_1 and σ_{n+1}. □

As a consequence, the points given in Theorem 4.4.3 must be in different orbits of B_{n+2}. We now show that the union of these orbits is the whole space $\mathbb{P}^n(\mathbb{Q})$. We first deal with the case $n = 3$, and the general case will follow by induction.

Proposition 4.4.7. *The action $B_5 \curvearrowright \mathbb{P}^3(\mathbb{Q})$ has two orbits, the one of $[1 : 0 : 0 : 1]$ and the one of $[1 : 0 : 0 : 0]$.*

Proof. First of all, by Proposition 4.4.6, the points $[1 : 0 : 0 : 0]$ and $[1 : 0 : 0 : 1]$ are in different orbits.
Let $p \in \mathbb{P}^3(\mathbb{Q})$. By Lemma 4.4.2, we can suppose that $p = [x_1 : 0 : x_3 : x_4]$. By the action of the right copy of B_3 inside B_5, we can even take $p = [x_1 : 0 : \lambda : 0]$, with $\lambda = \gcd(x_3, x_4)$. Next, we can consider the action of the left copy of B_4, and according to Theorem 4.2.4, reach the point $[a : k : a : 0]$, for $k = x_1 - \lambda$ and $a \in [\![0, k-1]\!]$, coprime with k.
We start an induction on k. If $k = 1$, then $a = 0$, and by applying $\sigma_2\sigma_1$, the point $[0 : 1 : 0 : 0]$ becomes $[1 : 0 : 0 : 0]$.
Suppose $k \geq 2$. Write the Euclidean algorithm for (k, a) : there exists $N \in \mathbb{N}$ and sequences $(r_i)_{0 \leq i \leq N+1}$, $(q_i)_{0 \leq i \leq N+1}$, such that

$$(r_0, r_1) := (k, a) \ , \ (r_N, r_{N+1}) = (1, 0),$$

$$r_i = r_{i+1}q_{i+2} + r_{i+2} \text{ and } 0 \leq r_{i+1} < r_i \text{ for } 0 \leq i \leq N-1.$$

The braid $\sigma_3^{q_{2i+1}}\sigma_4^{q_{2i}}$ sends the point $[a : k : r_{2i-1} : k - r_{2i-2}]$ to the point $[a : k : r_{2i+1} : k - r_{2i}]$. According to the parity of N, this means that the point $[a : k : a : 0]$ is in the same orbit as $[a : k : 1 : k - r_{N-1}]$ or $[a : k : 0 : k - 1]$. In the first case, apply $\sigma_4^{-r_{N-1}+1}$ to recover the second case $[a : k : 0 : k - 1]$.
Since $\gcd(a, k) = 1$, the left copy of B_3 can transform $[a : k : 0 : k - 1]$ in $[1 : 0 : 0 : k - 1]$, and then apply $\sigma_3\sigma_2\sigma_4\sigma_3$ to reach $[1 : k - 2 : 1 : 0]$.
By induction on k, this is in the same orbit of either $[1 : 0 : 1 : 0]$, either $[1 : 1 : 1 : 0]$. In the first case, apply $\sigma_3^{-1}\sigma_4^{-1}$ to get $[1 : 0 : 0 : 1]$. In the second case, apply $\sigma_1^{-1}\sigma_3$ and recover the base case of the induction. □

Now we can prove Theorem 4.4.3.

Proof of Theorem 4.4.3. Let $p \in \mathbb{P}^n(\mathbb{Q})$. Thanks to Lemma 4.4.2, we can suppose that the second coordinate of p is zero, $p = [x_1 : 0 : x_3 : x_4 : \cdots : x_{n+1}]$.
Let $\lambda := \gcd(x_3, x_4, \cdots, x_{n+1})$, and let $x'_i = x_i/\lambda$ for $i \in [\![3, n+1]\!]$. By induction, the point $[x'_3 : x'_4 : \cdots : x'_{n+1}] \in \mathbb{P}^{n-2}(\mathbb{Q})$ is in the orbit of $[1 : 0 : 0 : \varepsilon_1 : 0 : \varepsilon_2 : 0 : \cdots : \varepsilon_{m-1}]$ for some $\varepsilon_i \in \{0, 1\}$.
This means that by the action of the right copy of B_n, the point p can be changed to

$$[x_1 : 0 : \lambda : 0 : 0 : \lambda\varepsilon_1 : 0 : \lambda\varepsilon_2 : 0 : \cdots : \lambda\varepsilon_{m-1}].$$

Let the left copy of B_5 act on this point. By Proposition 4.4.7, the point $[x_1 : 0 : \lambda : 0]$ is in the orbit of $[1 : 0 : 0 : \varepsilon_m]$, for $\varepsilon_m \in \{0, 1\}$. Since $\gcd(x_1, \lambda) = 1$, we get

$$[1 : 0 : 0 : \varepsilon_m : 0 : \lambda\varepsilon_1 : 0 : \lambda\varepsilon_2 : 0 : \cdots : 0 : \lambda\varepsilon_{m-1}]$$

Finally, by the action of a copy of B_4, we can replace $[\varepsilon_m : 0 : \lambda\varepsilon_1]$ by $[\lambda\varepsilon_1 : 0 : \varepsilon_m]$, and then by Proposition 4.4.7, there is $\varepsilon'_{m-1} \in \{0, 1\}$ such that p is in the orbit of

$$[1 : 0 : 0 : \varepsilon'_{m-1} : 0 : \varepsilon_m : 0 : \lambda\varepsilon_2 : 0 : \cdots : 0 : \lambda\varepsilon_{m-1}].$$

Repeat this mouvement to reach a point whose coordinates are 0 or 1, and then sort the coordinates by actions of B_4 to reach the desired form. This shows that $\mathbb{P}^n(\mathbb{Q})$ is the union of orbits of the representatives given. □

4.4.2 Case of even n

For $n \in \mathbb{Z}_{>0}$ even, and $x = [x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n(\mathbb{Q})$, let us denote

$$\delta := \gcd(x_1 - x_3, x_2, x_3 - x_5, x_4, \cdots, x_{n-1} - x_{n+1}, x_n).$$

Theorem 4.4.8. *Let n be an even positive integer. Then the action $B_{n+2} \curvearrowright \mathbb{P}^n(\mathbb{Q})$ has infinitely many orbits. Orbit invariants are given by δ and $\pm x_1 \mod (\delta)$.*

Proof. It is straightforward to check that $\gcd(x_1 - x_3, x_2, x_3 - x_5, x_4, \cdots, x_{n-1} - x_{n+1}, x_n)$ does not change under the action of standard generators σ_k for $k \in [\![1, n+1]\!]$.
The coordinate x_1 is defined up to sign, and the value $x_1 \mod (\delta)$ does not change by the action of σ_1, hence it is preserved by the action of B_{n+2}, up to sign.
These invariants implies that there is an infinite number of orbits. For example, the following points are each in a different orbit :

$$p_{\delta,a} := [a : 0 : a + \delta : 0 : a : 0 : a + \delta : 0 : \cdots],$$

for $\delta \in \mathbb{N}^*$ and $a \in [\![1, \delta/2]\!]$, with $\gcd(a, \delta) = 1$. □

We conjecture a system of representatives for the case $n = 4$.

Conjecture 4.4.9. *Denote by $p_{\delta,a} = [a : 0 : a + \delta : 0 : a]$. A system of representatives for the action $B_6 \curvearrowright \mathbb{P}^4(\mathbb{Q})$ is given by*

$$\{p_{\delta,a} \mid \delta \in \mathbb{N}^*,\ a \in [\![0, \delta - 1]\!],\ \gcd(a, \delta) = 1\}.$$

Note that unlike the case of $n = 2$, we conjecture that the orbits of $p_{\delta,\delta-a}$ and $p_{\delta,a}$ are distinct, if $a \neq \delta - a$.

4.4.3 Proof of Lemma 4.4.2

Let $n \in \mathbb{N}^*$. We prove that for any point $p = [x_1 : x_2 : \cdots x_{n+1}] \in \mathbb{P}^n(\mathbb{Q})$, there is a braid $\beta \in B_{n+2}$ such that $\beta \cdot p$ has a zero second coordinate. To do this, we will rely on the following facts.

Fact 1 : if there is $k \in [\![0, n]\!]$ such that $x_{k-1} = x_{k+1}$, then the braid $\beta_k = \prod_{i=2}^{k} \sigma_i \sigma_{i-1}$ sends p to

$$[x_{k-1} + x_k : 0 : x_0 : x_1 : \cdots x_{k-1} : x_{k+2} : x_{k+3} : \cdots : x_n].$$

Let us call this braid the chain braid of index k.
Fact 2 : if $x_n = 0$, then the braid $\sigma_{n+1}\sigma_n$ changes p into $[x_1 : \cdots : x_{n-1} : x_{n+1} - x_{n-1} : x_{n-1}]$, which satisfies condition of Fact 1 with $k = n$.

Denote by k_0 the lowest index such that $|x_{k_0}| = \max(|x_i|, i \in [\![1, n+1]\!])$, and without loss of generality suppose that $x_{k_0} > 0$.
We proceed now by induction on x_{k_0}. If $x_{k_0} = 1$, then it is easy to check that p is in the same orbit as a point with a zero second coordinate.
Take $x_{k_0} > 1$.
Case 1 : $k_0 = n + 1$. Then either $x_n = 0$ and we are done by Fact 2, either $x_n \neq 0$ and then apply σ_{n+1}^e with $e = \operatorname{sign}(x_{n+1}) \operatorname{sign}(x_n)$ to replace x_{n+1} by $x_{n+1} - |x_{n-1}| < x_{n+1}$, and conclude by induction.
Case 2 : $k_0 = 1$. Then either $x_2 = 0$ and we are done, either $x_2 \neq 0$ and as before we can apply $\sigma_1^{\pm 1}$ and conclude by induction.
Case 3 : $2 \leq k_0 \leq n$. If $x_{k_0-1} = x_{k_0+1}$ we are done with Fact 1. Else, apply $\sigma_{k_0}^e$ with $e = \operatorname{sign}(x_{k_0}) \operatorname{sign}(x_{k_0-1} - x_{k_0+1})$, to replace x_{k_0} by $x_{k_0} - |x_{k_0+1} - x_{k_0-1}|$. Since $|x_{k_0-1}| < x_{k_0}$ and $|x_{k_0+1}| \leq x_{k_0}$, we have $|x_{k_0} - |x_{k_0+1} - x_{k_0-1}|| < x_{k_0}$, and we can conclude by induction.

Abrégé en français

État de l'art : un tour d'horizon de la littérature sur les q-nombres réels

Le concept de déformation quantique apparaît en mathématiques dans différents domaines, en théorie des représentations, en théorie des nœuds ou en calcul quantique, entre autres. L'idée est d'enrichir un objet en le plongeant dans une famille à un paramètre, ce paramètre étant traditionnellement noté q. L'évaluation $q \mapsto 1$ doit permettre de retrouver l'objet que l'on a déformé. Pour que la déformation soit intéressante, elle doit être pertinente vis-à-vis du contexte mathématique dans lequel évolue l'objet étudié. En combinatoire, des suites célèbres d'entiers naturels comme les factorielles ou les coefficients binomiaux possèdent des versions q-déformées naturelles, qui fournissent une énumération plus précise de ce que ces suites comptent classiquement. Ces déformations de suites d'entiers sont construites à partir des q-entiers, déjà utilisés par Euler et Gauss en leurs temps.

$$[n]_q := \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1},\ n \in \mathbb{N}. \tag{4.8}$$

Cette déformation des entiers naturels peut être étendue aux entiers relatifs en posant $[-n]_q = \frac{q^{-n}-1}{q-1} = -q^{-1}[n]_{q^{-1}}$. La combinatoire des q-entiers et des déformations affiliées est un domaine actif de recherche, et presque toutes les suites classiques d'entiers ont un analogue quantique, comme les q-coefficients binomiaux ou les q-nombres de Catalan.

Dans les années 2020, Sophie Morier-Genoud et Valentin Ovsienko ont proposé une généralisation des q-entiers à tous les nombres réels. Cela peut paraître surprenant qu'une telle généralisation n'ait pas vu le jour plus tôt, mais cela tient au fait que les idées naïves pour le faire ne sont pas satisfaisantes. En effet, remplacer l'entier n par un nombre rationnel x dans la formule (4.8) fait apparaître des puissances fractionnaires de q, ce qui ne se comporte pas bien d'un point de vue combinatoire. D'autre part, définir simplement $\left[\frac{a}{b}\right]_q = \frac{[a]_q}{[b]_q}$ aboutit à une notion de droite rationnelle déformée qui ne préserve pas l'ordre sur les réels, puisqu'en général, si $\frac{a}{b} < \frac{c}{d}$, le polynôme $[c]_q[b]_q - [a]_q[d]_q$ n'est pas à coefficients positifs.

La définition des q-nombres rationnels de Morier-Genoud et Ovsienko évite ces écueils, et s'est avérée particulièrement adaptée dans plusieurs contextes différents, des algèbres amassées à l'approximation Diophantienne, de la théorie des nœuds à l'algèbre homologique. Avant de présenter les résultats de cette thèse, nous présentons les grandes lignes de la littérature produite ces six dernières années sur les q-nombres réels. Ce sera l'occasion d'introduire les objets et les questions principales de cette théorie, dont nous approfondirons certains aspects dans le corps du manuscrit.

Construction des q-nombres réels

La définition la plus élémentaire des q-nombres réels utilise la représentation des réels en fractions continues. Pour tout $x \in \mathbb{R}$, il existe une unique suite d'entiers $(c_1, c_2, c_3, \cdots)$ avec $c_i \geq 2$ pour tout $i \geq 2$ (et telle qu'il n'y a pas de rang k tel que $c_i = 2$ pour tout $i \geq k$), telle que

$$x = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cfrac{1}{\ddots}}}.$$

Quand x est un nombre rationnel, la suite $(c_i)_i$ est finie, et la formule ci-dessus est bien définie. On note alors $x = [\![c_1, c_2, \cdots, c_k]\!]$, et on dit que cette écriture est la fraction continue négative, ou de Hirzebruch-Jung, de x. Si x est irrationnel, la formule ci-dessus est une notation qui signifie que la suite des réduites $(x_n = [\![c_1, c_2, \cdots, c_k]\!])_n$ converge vers x. L'analogue quantique de x est alors calculé via une déformation du développement en fraction continue, de la façon suivante [MGO20]

$$[x]_q = [c_1]_q - \cfrac{q^{c_1-1}}{[c_2]_q - \cfrac{q^{c_2-1}}{[c_3]_q - \cfrac{q^{c_3-1}}{\ddots}}}. \tag{4.9}$$

Ainsi, lorsque x est un nombre rationnel, sa déformation est une fraction rationnelle en q, qui s'évalue en x à $q = 1$. Lorsque x est un nombre irrationnel, la formule (4.9) est une notation signifiant que la suite des q-réduites $([x_n]_q = [\![c_1, c_2, \cdots, c_n]\!])_n$ converge en topologie q-adique vers une série de Laurent formelle [MGO22]. Dans le cas irrationnel, l'évaluation en $q = 1$ n'a donc a priori pas de sens avant d'étudier plus avant ces séries.

x classique	$\xrightarrow{q}$	$[x]_q$ quantique
$\mathbb{N}$		$\mathbb{N}[q]$
$\mathbb{Z}$		$\mathbb{Z}[q, q^{-1}]$
$\mathbb{Q}$		$\mathbb{Z}(q)$
$\mathbb{R}_{\geq 0}$		$\mathbb{Z}[[q]]$
$\mathbb{R}$		$\mathbb{Z}[[q]][q^{-1}]$

Cette déformation des nombres réels n'a pas été choisie arbitrairement. La raison principale qui explique sa pertinence est qu'elle munit les q-nombres réels d'une structure modulaire déformée, ce qui était suggéré dans [MGO20]. Leclere et Morier-Genoud ont en effet prouvé qu'il y a une action du groupe modulaire $\mathrm{PSL}_2(\mathbb{Z})$ sur la droite réelle déformée [LMG21], engendrée par

$$T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \text{ et } S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}.$$

En utilisant l'équivariance de la quantification par rapport à cette action déformée, elles ont déduit que la formule de déformation des fractions continues (4.9) donne le même résultat quel que soit le choix des coefficients c_i dans le développement en fraction continue.

Étonnamment, il se trouve qu'une autre déformation des nombres rationnels satisfait les mêmes propriétés d'équivariance vis-à-vis du groupe modulaire. Cela est dû au fait que l'opérateur T_q possède deux points fixes différents dans $\mathbb{P}^1(\mathbb{Z}(q))$, qui sont $\frac{1}{0}$ et $\frac{1}{1-q}$. La version jumelle des q-rationnels, correspondant au point fixe $\frac{1}{1-q}$, a été étudiée par Bapat, Becker et Licata [BBL23] et est appelée la version gauche. De ce fait, un nombre rationnel x possède deux q-analogues naturels, la version dite droite, notée $[x]_q^\sharp$ et calculée par (4.9), et la version gauche, notée $[x]_q^\flat$, calculée par une légère modification de la formule de fraction continue déformée :

$$[x]_q^\flat = [c_1]_q - \cfrac{q^{c_1-1}}{[c_2]_q - \cfrac{q^{c_2-1}}{\ddots \cfrac{}{[c_{k-1}]_q - \cfrac{q^{c_{k-1}-1}}{[c_k]_q^\flat}}}}, \tag{4.10}$$

avec $[n]_q^\flat = \frac{q^{n+1}-q^n+q^{n-1}-1}{q-1} = 1 + q + \cdots + q^{n-2} + q^n$ pour $n \in \mathbb{N}$.

Asymptotiquement, la version gauche et la version droite coïncident, les nombres irrationnels n'ont donc qu'un seul analogue quantique qui est la limite commune des versions gauches et droites.

Combinatoire des q-rationnels

Grâce aux symétries modulaires, les analoques quantiques de nombres rationnels ont de très bonnes propriétés combinatoires. Nous listons ci-dessous différents modèles qui les décrivent. Cette liste n'est pas exhaustive, mais a pour but de donner un large survol de la riche combinatoire des q-rationnels. Certains de ces modèles seront détaillés plus amplement dans le corps du document.

- Graphe de Farey : les q-rationnels peuvent être calculés en utilisant une version déformée du graphe de Farey, construite pour la version droite dans [MGO20]. C'est un outil efficace, utilisé notamment pour montrer la positivité totale : si deux nombres sont ordonnés $\frac{a}{b} < \frac{c}{d}$, alors leurs déformations droites $\frac{A}{B}$ et $\frac{C}{D}$ satisfont $BC - AD \in \mathbb{Z}_{>0}[q]$. Ce modèle fournit également une interprétation combinatoire des q-rationnels en termes de chemins dans une version orientée du graphe de Farey. La version gauche a été traitée dans [FQ25, Ren24].

- Posets zigzags ou carquois linéaires : par une correspondance entre les chemins dans le graphe de Farey et les idéaux ordonnés dans les posets zigzags, on peut interpréter les q-rationnels comme fonctions génératrices de ces derniers. Cela a été montré (dans le langage des clôtures dans les carquois de type A) par Morier-Genoud et Ovsienko [MGO20]. Ce point de vue est l'outil principal pour montrer la conjecture d'unimodalité, stipulant que les numérateurs et dénominateurs des q-rationnels en version droite sont unimodaux. Les premiers pas vers cette conjecture ont été fait par McConville, Sagan et Smyth [MSS21], puis une preuve complète a été établie par Kantarcı Oğuz et Ravichandran, grâce à l'utilisation de posets zigzags circulaires [KOR23]. Dans le même mouvement, ils prouvent que les fonctions génératrices de posets zigzags circulaires sont symétriques, ce qui correspond à la propriété de palindromicité des traces de matrices q-déformées [LMG21]. Si la preuve de [KOR23] était

récursive, Elizalde et Sagan ont trouvé plus tard une preuve bijective de la symétrie des fonctions génératrices de posets zigzags circulaires [ES23]. L'unimodalité des fonctions génératrices de posets zigzags s'inscrit dans le cadre plus général des F - polynômes d'algèbres amassées venant de surfaces, comme l'ont montré Kang, Lee et Lim [KLL26].

- Graphes en serpents : comme l'ont montré Çanakçı, Schiffler [CS20], la combinatoire des fractions continues peut être traduite dans le modèle des graphes en serpents. Ce point de vue a été utilisé par Ovenhouse pour apporter une nouvelle approche aux q-rationnels droits, en termes de nombres de points dans les cellules de Schubert sur des corps finis [Ove23]. À notre connaissance, c'est la seule connexion (pour le moment !) entre les q-rationnels et la géométrie sur les corps finis. Dans un autre contexte, Burcroff, Ovenhouse, Schiffler et Zhang ont proposé une généralisation en dimension supérieure des q - rationnels droits [BOSZ24] en considérant des appariements parfaits généralisés dans des graphes en serpents, mettant à profit la dualité entre ces appariements dans un graphe en serpents et les chemins nord-est généralisés dans un autre graphe en serpents (le dual). Ils ont prouvé une version généralisée de la positivité totale et des propriétés de convergence des q-rationnels. Finalement, un nouveau modèle pour les q-rationnels droits a été introduit par Ovsienko, se fondant sur des comptages d'appariements parfaits avec poids dans des graphes en serpents [Ovs25].

- Frises : peu après l'apparition des q-rationnels, Morier-Genoud et Ovsienko ont appliqué leur nouvelle théorie aux frises de Conway-Coxeter. Ils ont défini une notion de frise q-déformée, et montré dans le contexte quantique des résultats combinatoires analogues aux résultats classiques sur les frises [MGO21]. Cette notion de q-frises a récemment été étudiée en détail et légèrement étendu par Inuma, Motomiya et Kogiso [IMK26].

- Partitions hyperbinaires : McConville, Propp et Sagan ont mis en évidence un nouvel isomorphisme entre le treillis des partitions hyperbinaires d'un entier n et le treillis des idéaux ordonnés d'un poset zigzag associé [MPS26]. Leur travail relie la combinatoire des partitions hyperbinaires à la théorie des nombres déformés, et ils en déduisent de nouvelles méthodes de calcul pour manipuler les q-rationnels.

- Système de numération d'Ostrowski : ce dernier modèle a été mis en évidence par Aval et Labbé, qui considèrent une version modifiée du système de numération d'Ostrowski, à partir d'une notion de suite admissible attachée à un développement en fraction continue [AL25]. Ils montrent que les q-rationnels peuvent être interprétés comme les fonctions génératrices de telles suites admissibles. Cela les mène à définir une version améliorée des modèles de posets zigzags et de graphes en serpents associés aux nombres rationnels déformés droits, qui leur permet de décrire tous les rationnels positifs et pas seulement les rationnels supérieurs à 1 comme auparavant.

Aspects analytiques des q-nombres irrationnels

Avec la définition de Morier-Genoud et Ovsienko [MGO22], l'analogue quantique d'un nombre irrationnel est une série formelle en q. Il est alors naturel de s'intéresser au rayon de convegence de cette série, et si une forme close peut être calculée. Cela conduit

à considérer q non plus comme une variable formelle, mais comme un paramètre réel ou complexe. Les premiers pas dans cette direction ont été fait par Morier-Genoud et Ovsienko pour les nombres quadratiques, et ce cas a été entièrement résolu par Leclere et Morier-Genoud [LMG21]. Ainsi, pour $x = \frac{a\pm\sqrt{d}}{b}$ un nombre quadratique, sa déformation est solution d'une équation polynomiale de degré 2, et peut donc être écrite sous la forme

$$\left[\frac{a \pm \sqrt{d}}{b}\right]_q = \frac{A \pm \sqrt{D}}{B}, \tag{4.11}$$

où A, B et D sont trois polynômes de Laurent en q. La clé de la preuve est l'action équivariante du groupe modulaire sur les q-nombres. De plus, en étudiant les traces des matrices déformées qui définissent cette action, elles ont montré que le polynôme D est palindromique.
Pour un nombre quadratique x, le rayon de convergence de la série $[x]_q$ dépend donc seulement des racines des polynômes B et D. Ces racines ont été étudiées par Leclere, Morier-Genoud, Ovsienko et Veselov dans le cas particulier du nombre d'or φ et du nombre d'argent $\sqrt{2}$, en calculant explicitement leurs rayons de convergence [LMGOV24]. Ils conjecturent que parmi tous les nombres réels, $[\varphi]_q$ a le rayon de convergence le plus bas, qui est $R_\star = \frac{3-\sqrt{5}}{2} \simeq 0.38$. Ils se sont attelés à démontrer cette conjecture en commençant par les nombres rationnels. Sous certaines hypothèses sur $x \in \mathbb{Q}$, ils montrent que le rayon de convergence de $[x]_q^\sharp$ est en effet plus grand que $R_\star$. Ils obtiennent également un résultat plus faible valable pour tout nombre rationnel : le rayon de convergence de $[x]_q^\sharp$ est plus grand que $3 - 2\sqrt{2} \simeq 0.17$ pour tout $x \in \mathbb{Q}$. Leur preuve consiste à encadrer les racines des numérateurs et des dénominateurs des rationnels quantiques, montrant qu'elles sont contenues dans l'anneau $\{3 - 2\sqrt{2} < |q| < 3 + 2\sqrt{2}\}$. Ce théorème a été étendu aux nombre irrationnels quadratiques par Leclere dans sa thèse [Lec24]. Mentionnons que ce fait a été ensuite utilisé dans [MGOV24] pour étudier les spécialisations de la représentation de Burau du groupe de tresse B_3.

Quelques mois plus tard, Ren a démontré la conjecture dans le cas des nombres métalliques [Ren24], une famille de nombres quadratiques dont le développement en fraction continue est de la forme $[n, n, n, \cdots]$ pour $n \in \mathbb{Z}_{\geq 1}$. Le premier nombre métallique est le célèbre nombre d'or. Le deuxième est le nombre d'argent $\sqrt{2}$, et il a été montré dans [LMGOV24] que le rayon de convergence de $[\sqrt{2}]_q$ est supérieur à $R_\star$. En plus du résultat pour tous les nombres métalliques, Ren a aussi prouvé que la conjecture tient pour toutes les réduites $[n, n, \cdots, n]$.

En 2024, la conjecture de [LMGOV24] a été montrée pour tous les nombres rationnels droits par Elzenaar, Gong, Martin et Schillewart, grâce à des méthodes géométriques [EGMS25]. Complétant ce résultat, Evans, Veselov et Winn ont étudié les q-nombres rationnels dont le rayon de convergence est maximal (égal à 1), qu'ils appellent les fractions de Kronecker [EVW24].

Finalement, en 2026, Etingof a trouvé une région ouverte $\mathcal{D}$, contenant le disque de rayon $3 - 2\sqrt{2}$, dans laquelle tout q-nombre réel converge vers une fonction holomorphe qui ne s'annule pas [Eti25, Théorème 1.1]. Il a montré que cela se produit également à l'intérieur du disque de rayon $2 - \sqrt{3} \simeq 0.27$. Les deux théorèmes sont complémentaires, voir la

figure ci-dessous. De plus, il a déduit que pour tout nombre réel x, la fonction $q \mapsto [x]_q$ est bien définie et analytique sur l'intervalle ouvert $]-R_\star, 1[$, renforçant la conjecture de [LMGOV24]. En corollaire, cela donne un sens à la spécialisation en $q = 1$ de $[x]_q$, pour x irrationnel.

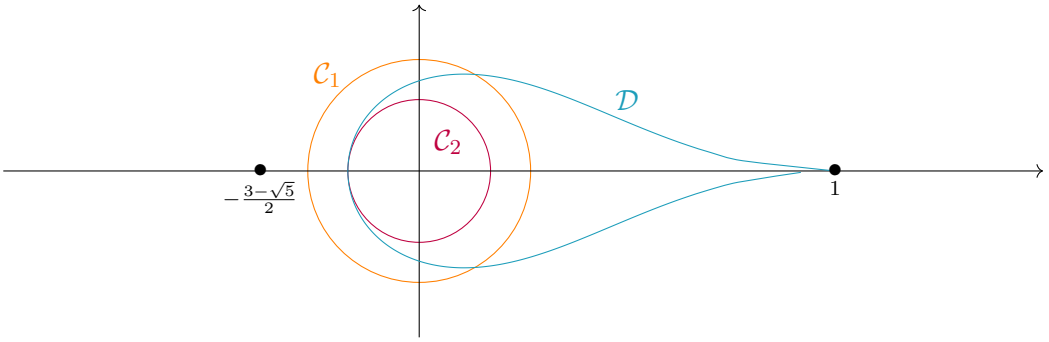


Figure 4.3: La région ouverte $\mathcal{D}$ (en bleu) dans laquelle tout q-nombre réel $[x]_q$ converge. $\mathcal{C}_1$ (en orange) est le cercle de rayon $2-\sqrt{3}$, dans lequel il y a aussi convergence de tout q-réel. $\mathcal{C}_2$ (en rose) est le cercle de rayon $3-2\sqrt{2}$.

Mis à part le rayon de convergence de $[x]_q$, d'autres travaux intéressants ont été mené sur l'analyse des q-nombres réels. Dans [Las25], Lasker étudie la fonction d'une variable réelle $q \mapsto [x]_q^\sharp$ autour de $q = 1$, pour $x \in \mathbb{Q}$. Par construction, $[x]_q^\sharp(q = 1) = x$, mais Lasker calcule une formule explicite pour les dérivées premières et secondes de cette fonction, en $q = 1$, faisant un lien avec la fonction de Thomae. La preuve suit une récurrence astucieuse dans le graphe de Farey avec poids. La fonction $q \mapsto [x]_q$ a également été étudiée par Etingof, pour tout réel x et pour la variable q contenue dans le rayon de convergence de la série [Eti25, Proposition 5.3]. En particulier, il a prouvé que la dérivée (par rapport à q) de $[x]_q$ est positive pour $0 < q < 1$.
D'autre part, la fonction $x \mapsto [x]_q$ est strictement croissante sur $\mathbb{R}$, pour $0 < q < 1$. Sur les nombres rationnels, c'est une conséquence de la positivité totale, et pour les nombres irrationnels cela a été formulé par Bapat, Becker et Licata pendant leur étude des q - rationnels gauches. Leur approche implique en effet que pour tous les nombres rationnels $x < y$, les déformations de x et y sont bien ordonnées

$$[x]_q^\flat < [x]_q^\sharp < [y]_q^\flat < [y]_q^\sharp.$$

Cela explique d'ailleurs les noms des q-rationnels droits et gauches. Bapat, Becker et Licata ont prouvé que pour un paramètre fixé $q \in]0, 1[$, la fonction $x \mapsto [x]_q^\sharp$ est continue à droite sur $\mathbb{R}$. Plus précisément, ils montrent que pour toute suite $(x_n)_n$ de rationnels convergeant vers $x \in \mathbb{R}$, quatre cas sont à distinguer (cette situation avait déjà été observée par Morier-Genoud et Ovsienko, voir la remarque 3.2 dans [MGO22]) :

1. si $x_n > x$ pour tout n, et $x \in \mathbb{Q}$, alors $[x_n]^\sharp \underset{n\mapsto+\infty}{\longrightarrow} [x]_q^\sharp$;
2. si $x_n < x$ pour tout n, et $x \in \mathbb{Q}$, alors $[x_n]^\sharp \underset{n\mapsto+\infty}{\longrightarrow} [x]_q^\flat$;
3. si $x \in \mathbb{R} \setminus \mathbb{Q}$, alors $[x_n]^\sharp \underset{n\mapsto+\infty}{\longrightarrow} [x]_q$;
4. sinon, la suite $([x_n]_q^\sharp)_n$ ne converge pas.

Des énoncés analogues tiennent pour des suites de q-rationnels gauches, voir la Section 1.5. Ce résultat a été étendu par Etingof pour n'importe quel paramètre $q \in \mathcal{D}$. De plus, pour q réel dans l'intervalle $]0,1[$, il démontre que la fonction $x \mapsto [x]_q$ est différentiable avec une dérivée nulle presque partout sur $\mathbb{R}$.

Du point de vue de la théorie des nombres, les coefficients des séries $[x]_q$, pour x irrationnel, présentent également un intérêt. Cela a été étudié par Ovsienko, Pedon [OP25] puis Han, Pedon [HP25] dans le cas des nombres métalliques. Ils ont notamment montré des propriétés remarquables de périodicité sur les déterminants de Hankel de ces séries, ainsi que des relations de récurrence de type Somos-Gale-Robinson. Très récemment, Pedon a établi de nombreuses identités remarquables satisfaites par les coefficients des q-nombres métalliques, leur comportement asymptotique, une croissance logarithmique et un lien avec les structures secondaires de l'ARN [Ped26].

Une autre question naturelle autour des q-irrationnels concerne le comportement des déformations de nombres algébriques. Le cas des irrationnels quadratiques, entièrement résolu dans [LMG21], donne l'espoir d'une théorie cohérente, mais la généralisation à des degrés plus grands s'est avérée être un problème difficile. Ovsienko et Ustinov ont travaillé sur des exemples de nombres cubiques [OU25]. Dans le Chapitre 1 de ce document, nous considérons des familles de nombres algébriques de degrés 4 et 6.

Pour finir, Etingof a établi une notion de nombre complexe q-déformé, grâce à un prolongement méromorphe de la fonction de quantification des réels [Eti25, Section 7].

q-Nombres réels en théorie des nœuds

L'une des origines de la théorie des q-réels tient au lien entre les algèbres amassées et les nœuds rationnels, lien qui a été établi par Lee et Schiffler en 2017 [LS19]. Le même type de combinatoire avait également été remarquée par Kogiso et Wakui lorsqu'ils ont décrit un pont entre les frises et les nœuds rationnels [KW19]. C'est cette même combinatoire qui sous-tend en fait la définition des q-rationnels [MGO20]. Cela a été trouvé indépendamment par Sikora lors de son travail sur la conjecture de Jones ("Jones Unknot conjecture"), dans le cas particulier des enchevêtrements rationnels [Sik24]. Il prouve notamment que le crochet de Kauffman peut-être utilisé pour calculer des q-rationnels (ou inversement, on peut utiliser les q-rationnels pour calculer des crochets de Kauffman). Quelques articles ont ensuite exploité le lien entre les q-nombres et le polynôme de Jones [Wak25, KMR$^+$25, RY25, BRY26].

Sur un sujet différent, la célèbre représentation de Burau réalise un lien entre la théorie des tresses et les q-rationnels, comme l'observent Morier-Genoud, Ovsienko et Veselov [MGOV24]. Dans cet article, les trois auteurs exploitent ce lien pour identifier quelles spécialisations de la représentation de Burau de B_3 sont fidèles. La fidélité de la représentation de Burau est une question importante en théorie des nœuds. On sait depuis longtemps que les représentations de B_2 et B_3 sont fidèles. A l'inverse, après un long travail, Moody [Moo91] puis Long, Paton [LP93] et finalement Bigelow [Big99] ont pu démontrer que pour tout $n \geq 5$, la représentation de B_n n'est pas fidèle. Il reste donc le cas de B_4, qui est justement relié à la conjecture de Jones. Lors de cette thèse, nous nous intéressons à une action naturelle de B_4 sur le plan rationnel, donnée par la représentation

de Burau spécialisée à $t = -1$ [Jou25a]. Ce travail est présenté dans le Chapitre 4.

Très récemment, une nouvelle connexion entre les q-rationnels et la théorie des nœuds a été implémentée [JQ26], en construisant une famille d'invariants d'entrelacs indexée par $x \in \mathbb{Q}$. Ces invariants ne sont pas inédits, puisqu'ils correspondent en fait à une spécialisation du polynôme de HOMFLY-PT. Ils ont cependant le mérite de fournir une interpolation cohérente aux invariants de Reshetikhin-Turaev, paramétrés par les entiers $n \in \mathbb{N}^*$.

Arithmétique et q-nombres réels

La théorie des déformations de nombres réels a évidemment des intersections avec la théorie des nombres. Le premier lien remarquable se fait avec les nombres de Markov, une famille célèbre de solutions de l'équation diophantienne

$$x^2 + y^2 + z^2 = 3xyz.$$

Plusieurs q-analogues de ces nombres existent (pour davantage de détails, voir le préambule du Chapitre 2). Les q-nombres rationnels offrent une déformation naturelle, qui a suscité beaucoup de publications [Kog20, LL22, KO25, LLS24, EJMGO26].

Dans la même idée, Morier-Genoud, Ovsienko et Mathevet ont travaillé récemment sur une déformation des triplets pythagoriciens [MMGO26], en introduisant une équation de Pythagore déformée et construisant une classe de solutions satisfaisant certaines conditions combinatoires.

Des outils de la théorie des nombres ont également été appliqué aux q-rationnels, par exemple par Uludağ et Yılmaz qui ont étudié des généralisations des fonctions numérateur et dénominateur, retrouvant les q-rationnels dans un cas particulier [UY22].

Le q-nombre réel $[x]_q$ est une alternative pertinente à l'expression $\frac{q^x-1}{q-1}$ dans le domaine du calcul quantique, comme l'ont mis en évidence Machacek et Ovenhouse [MO24]. Ils montrent que de nombreuses identités quantiques sont satisfaites par les q-nombres réels, et en particulier ils définissent et étudient un analogue de la fonction Gamma.

Grâce à une étude précise des fractions continues et de l'interprétation en posets zigzags des q-rationnels, Kogiso, Miyamoto, Ren, Wakui et Yanagawa démontrent dans [KMR+25] une condition suffisante pour que les dénominateurs de deux q-rationnels soient identiques. Ils caractérisent aussi les numérateurs et dénominateurs de q-rationnels qui sont des palindromes.

Le groupe modulaire et ses généralisations

Puisque la déformation de la droite réelle passe par la déformation de l'action du groupe modulaire $\mathrm{PSL}_2(\mathbb{Z})$, il est très tentant de généraliser ce processus pour déformer les nombres complexes, en étendant le groupe à $\mathrm{PSL}_2(\mathbb{Z}[i])$. Cette idée a été explorée par Ovsienko dans [Ovs21], où il étudie un q-analogue de la translation imaginaire $z \mapsto z+i$, déterminée uniquement par sa compatibilité avec l'action déformée du groupe modulaire. Utilisant cela, il définit une version déformée des entiers de Gauss et en donne des formules explicites

en lien avec les polynômes de Chebyshev.

L'action déformée de $\mathrm{PSL}_2(\mathbb{Z})$ sur la droite réelle a aussi été prise comme point de départ par Alexander Thomas [Tho24] pour définir une q-version de l'algèbre de Lie $\mathfrak{sl}_2(\mathbb{R})$, c'est-à-dire une algèbre de Lie isomorphe à $\mathfrak{sl}_2(\mathbb{R})$, mais définie sur l'anneau des polynômes en q. Il interprète les éléments de $\mathfrak{sl}_2(\mathbb{R})$ comme des opérateurs différentiels, et en considérant le q-analogue de l'opérateur $x\partial$, il trouve une transformation de Möbius appelée l'application de transition, comblant l'écart entre les versions gauches et droites des q-rationnels. Dans ce contexte, il définit également une déformation des algèbres de Heisenberg et de Witt.

La question de la spécialisation des matrices q-déformées a été soulevée dès l'article fondateur de Morier-Genoud et Ovsienko [MGO20], où il est montré qu'en $q = -1$, le q-groupe modulaire est réduit au groupe symétrique $\mathfrak{S}_3$. Dans son manuscrit de thèse [Lec24], Leclere a travaillé sur les évaluations de q aux racines de l'unité, et ses résultats ont été énoncé indépendamment plus tard par Byakuno, Ren et Yanagawa [BRY26]. Le résultat, en bref, est que le groupe modulaire q-déformé est fini si et seulement si q est une racine $n$ième de l'unité, pour $n \in [\![2, 5]\!]$, et dans ce cas il est réduit à un groupe d'isométries de polygones réguliers.

Par ailleurs, une notion de q-transposée compatible avec les matrices déformées a été définie par Ren et Yanagawa [RY25]. Grâce à cet outil, ils donnent une autre preuve de résultats arithmétiques de [KMR+25], et les étendent à la version gauche des q-rationnels.

Le q-groupe modulaire peut être compris comme une représentation fidèle de $\mathrm{PSL}_2(\mathbb{Z})$ dans $\mathrm{PGL}_2(\mathbb{Z}[q, q^{-1}])$. Dans le Chapitre 1, nous verrons que c'est en fait la seule telle représentation, à conjugaison près. Si l'on autorise des coefficients complexes, alors l'unicité échoue, comme le montrent Topkara et Uludağ dans [TU25]. Ils trouvent en effet deux représentations fidèles en travaillant sur l'anneau $\mathbb{Z}[\sigma][q, q^{-1}]$ où σ est une racine primitive 6ième de l'unité.

Interprétations en algèbre homologique

Nous avons déjà discuté du travail de Bapat, Becker et Licata, comme la première étude de la version gauche des q-rationnels. Cependant, l'accent principal de leur article [BBL23] consiste à décrire les compactifications de l'espace de conditions de stabilité d'une catégorie de type A_2. Ces compacitifications sont paramétrées par un réel positif q, et pour chaque valeur de q, ils montrent qu'il y a un morphisme ρ_q entre la frontière de l'espace q - compactifié et la frontière du pavage de Farey q-déformé dans le plan hyperbolique. Au cours de leur preuve, ils déforment une bijection (déjà connue) entre les objets sphériques de la catégorie et les nombres rationnels. Étant donné un objet sphérique X associé au nombre $x \in \mathbb{Q}$, ils considèrent des données algébriques attachées à X, et calculent à partir de cela la préimage par ρ_q de la géodésique entre $[x]_q^\flat$ et $[x]_q^\sharp$ dans la frontière du pavage de Farey q-déformé. De plus, il y a une action du groupe de tresses B_3 sur l'espace q - compactifié. Ils prouvent que le morphisme ρ_q est équivariant par rapport à cette action et à l'action déformée de $\mathrm{PSL}_2(\mathbb{Z})$ sur le pavage de Farey q-déformé.

Les aspects combinatoires de ce travail ont été exploré par Ren dans [Ren24], où il étudie une interprétation de la q-somme de Farey dans le cadre homologique, et donne une nouvelle preuve, utilisant uniquement des méthodes combinatoires, d'un résultat ap-

paraissant dans [BBL23] à propos des q-rationnels gauches.

Ce contexte d'algèbre homologique a aussi fait l'objet d'un travail par Fan et Qiu, qui ont construit une réalisation topologique des q-rationnels, montrant qu'un nombre rationnel déformé peut être calculé en utilisant une notion de q-intersection d'arcs dans un disque décoré bigradué [FQ25]. De plus, ils considèrent une catégorification de leur modèle topologique, retrouvant les résultats de Bapat, Becker, Licata et les inscrivant dans un cadre plus général.

Résumé du contenu de la thèse

La visée générale de cette thèse consiste à étendre la construction des nombres déformés de Morier-Genoud et Ovsienko, et d'explorer les nombreux liens entre cette théorie et d'autres champs des mathématiques. Plusieurs directions complémentaires de généralisation ont été étudiées, en abordant la théorie des q-nombres sous différents angles.

La première direction s'intéresse aux symétries de la quantification, portées par l'action du groupe modulaire. Ces symétries régissent les propriétés algébriques des q-nombres. Nous construisons dans le Chapitre 1 une double extension de cette action, de laquelle nous déduisons des nouvelles relations entre les deux versions (gauche et droite) des q-nombres rationnels. Cette meilleure compréhension des symétries des q-nombres réels nous permet également d'étudier le comportement des déformations de certains nombres algébriques, dans la continuité des travaux antérieurs sur les nombres quadratiques et cubiques.

La deuxième direction envisagée est de nature combinatoire. On peut en effet dériver de la déformation des nombres rationnels des analogues de suites ou d'ensembles de nombres classiques. C'est le même principe derrière la déformation des factorielles et des coefficients binomiaux, qui sont construits à partir des q-entiers sans leur être identiques. Le Chapitre 2 présente ainsi une déformation des nombres de Markov, construite en s'appuyant sur les traces des matrices quantifiées attachées aux q-nombres rationnels. Les résultats principaux de ce chapitre ont été établis lors d'un travail collaboratif avec Sam Evans, Sophie Morier-Genoud et Valentin Ovsienko.

Un troisième axe concerne la géométrie des q-nombres réels. La motivation initiale de ce travail était de définir une extension de l'application de déformation aux nombres complexes, ce qui semble être un problème difficile à résoudre avec des méthodes purement algébriques. Dans le Chapitre 3, nous introduisons donc une nouvelle interprétation des q-nombres réels en tant qu'objets géométriques, dans l'espoir que cette démarche fournira le cadre adapté à une déformation du plan complexe. Notre étude se concentre pour le moment sur la droite réelle, où la déformation d'un nombre rationnel donne lieu à un disque, dépendant du paramètre de quantification q (vu comme une variable réelle). L'un des apports importants de ce point de vue est la description de nouvelles opérations entre q-rationnels, que nous appelons les opérations de Springborn. Ce chapitre est entièrement le fruit d'un travail commun avec Olga Paris-Romaskevich et Alexander Thomas (qui est l'initiateur de ce projet).

La quatrième et dernière thématique étudiée dans cette thèse a trait à la généralisation de la théorie de Morier-Genoud et Ovsienko en dimension supérieure. Nous choisissons, dans le Chapitre 4, d'exploiter la correspondance existante entre les analogues quantiques de nombres réels et la représentation de Burau du groupe de tresse B_3, pour définir une déformation du plan rationnel à partir de la représentation de Burau du groupe de tresse

B_4. À cette fin, nous classifions les orbites du plan sous l'action de B_4, et nous donnons une description des stabilisateurs de cette même action. La déformation du plan qui découle de cette approche est particulièrement riche, et appelle de plus amples travaux pour comprendre sa structure.

Dans cet abrégé en français, nous exposons succintement les idées et les résultats principaux de chaque chapitre. Afin de faciliter la navigation, les numérotations coïncident entre l'abrégé en français et le corps du manuscrit en anglais. Une partie des travaux exposés dans cette thèse a déjà été présentée dans les articles suivants.

[Jou25b] P. Jouteur. Symmetries of the q-deformed real projective line. *arXiv*, 2503.02122, 2025.

[EJMGO26] S. Evans, P. Jouteur, S. Morier-Genoud, V. Ovsienko. On q-deformed Markov numbers. Cohn matrices and perfect matchings with weighted edges. *Annals of Combinatorics*, 2026.

[JPRT26] P. Jouteur, O. Paris-Romaskevich, A. Thomas. Plane geometry of q-rationals and Springborn operations. *arXiv*, 2603.04295, 2026.

[Jou25a] P. Jouteur. Burau representation of B_4 and quantization of the rational projective plane. *Comptes Rendus. Mathématiques,* 363:89-107, 2025.

Abrégé du Chapitre 1 : Analogues quantiques de nombres

Ce chapitre introduit les objets principaux de la thèse, les analoques quantiques de nombres. Nous utilisons une définition reposant sur le groupe modulaire, dont une présentation par générateurs et relations est $\mathrm{PSL}_2(\mathbb{Z}) = \langle T, S \mid S^2 = (TS)^3 = 1\rangle$, avec

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Pour toute la suite, on considère une variable formelle notée q.

La déformation des nombres rationnels se fait par l'action déformée du groupe modulaire sur la droite projective $\mathbb{P}^1(\mathbb{Q})$. Cette action déformée est engendrée par les deux q-matrices suivantes.

$$T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Z}[q, q^{-1}]).$$

Un résultat de Leclere et Morier-Genoud [LMG21] assure que le groupe engendré par T_q et S_q est isomorphe à $\mathrm{PSL}_2(\mathbb{Z})$, ce qui confère à chaque matrice $M \in \mathrm{PSL}_2(\mathbb{Z})$ un analogue $M_q \in \langle T_q, S_q\rangle$.
On notera $\mathrm{PSL}_{2,q}(\mathbb{Z}) = \langle T_q, S_q\rangle$. Les q-rationnels sont alors donnés par les orbites de $\infty = \frac{1}{0}$ et $\frac{1}{1-q}$ sous l'action de $\mathrm{PSL}_{2,q}(\mathbb{Z})$ par homographie.

Définition 1.2.10 ([MGO20, BBL23])**.** Soit $x \in \mathbb{Q}$ et soit $M \in \mathrm{PSL}_2(\mathbb{Z})$. On suppose que $M \cdot \frac{1}{0} = x$. Les déformations droite et gauche de x sont des éléments de $\mathbb{P}^1(\mathbb{Z}(q))$ donnés par

- $[x]_q^\sharp = M_q \cdot \frac{1}{0}$ (version droite),
- $[x]_q^\flat = M_q \cdot \frac{1}{1-q}$ (version gauche).

Exemple 1.2.12. La version droite généralise les célèbres q-entiers d'Euler et Gauss. Pour tout $n \in \mathbb{Z}_{\geq 0}$,

$$[n]^\sharp = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}. \tag{1.12}$$

La version gauche, quant à elle, produit des nouveaux analogues d'entiers :

$$[n]^\flat = \frac{q^{n+1} - q^n + q^{n-1} - 1}{q - 1} = 1 + q + \cdots + q^{n-2} + q^n. \tag{1.13}$$

Une conséquence directe de la définition des q-rationnels est la propriété d'équivariance sous l'action du groupe modulaire.

Corollaire 1.2.14 ([LMG21, BBL23])**.** *Pour tout $M \in \mathrm{PSL}_2(\mathbb{Z})$, pour tout $x \in \mathbb{P}^1(\mathbb{Q})$,*

$$M_q \cdot [x]^\sharp = [M \cdot x]^\sharp, \text{ and } M_q \cdot [x]^\flat = [M \cdot x]^\flat.$$

Le choix des matrices T_q et S_q pour déformer l'action du groupe modulaire n'est pas arbitraire. La matrice T_q est imposée par la relation $[x+1]_q = q[x]_q + 1$, fondamentale dans la plupart des utilisations des q-entiers. De là, si l'on veut travailler avec des polynômes de Laurent en q, il n'y a que deux possibilités pour la matrice S_q.

Théorème 1.2.17. *Soit* $\rho : \mathrm{PSL}_2(\mathbb{Z}) \longrightarrow \mathrm{PGL}_2(\mathbb{Z}[q, q^{-1}])$ *une représentation, telle que pour tout* $M \in \mathrm{PSL}_2(\mathbb{Z})$*, on ait* $\rho(M)_{|q=1} = M$*. Supposons que* $\rho(T) = T_q$*. Alors* $\rho(S)$ *est donnée par l'une des deux matrices suivantes*

$$S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} \ , \ \widehat{S_q} = \begin{pmatrix} 1-q & -1 \\ q^2-q+1 & q-1 \end{pmatrix}.$$

De plus, les deux représentations sont conjuguées dans $\mathrm{PGL}_2(\mathbb{Z}(q))$*, par la matrice*

$$U_q = \begin{pmatrix} q & 1-q \\ 0 & q^2-q+1 \end{pmatrix}.$$

Ainsi, à conjugaison près, la déformation de Morier-Genoud et Ovsienko est la seule raisonnable. Nous allons l'étendre à $\mathrm{PGL}_2(\mathbb{Z})$, puis à une extension d'indice 2 de $\mathrm{PGL}_2(\mathbb{Z})$. Pour cela, nous avons besoin d'augmenter un peu l'anneau des coefficients sur lequel nous travaillons.

Notation 1.3.10. Soit $t_q = q^2 - q + 1$, et Λ la localisation par rapport à t_q dans $\mathbb{Z}[q, q^{-1}]$.

$$\Lambda := \mathbb{Z}[q, q^{-1}][t_q^{-1}].$$

L'anneau Λ est intègre, son corps des fractions est $\overline{\Lambda} = \mathbb{Z}(q)$, et ses unités sont données par $\{\pm q^i t_q^j \mid i, j \in \mathbb{Z}\}$.

Définition 1.3.2 ([Jou25b])**.** Posons $N_q = \begin{pmatrix} -1 & 1-q^{-1} \\ q-1 & 1 \end{pmatrix} \in \mathrm{PGL}_2(\Lambda)$, et notons $\mathrm{PGL}_{2,q}(\mathbb{Z})$ le groupe engendré par T_q, S_q et N_q, dans $\mathrm{PGL}_2(\Lambda)$.

Proposition 1.3.3 ([Jou25b])**.** *Les éléments* T_q*,* S_q *et* N_q *vérifient les relations suivantes dans* $\mathrm{PGL}_2(\Lambda)$ *:*

$$(T_q S_q)^3 = S_q^2 = N_q^2 = (N_q T_q)^2 = (N_q S_q)^2 = 1.$$

Le groupe $\mathrm{PGL}_{2,q}(\mathbb{Z})$ *est donc isomorphe à* $\mathrm{PGL}_2(\mathbb{Z})$*.*

Cette déformation de l'action de $\mathrm{PGL}_2(\mathbb{Z})$ est la seule qui soit compatible avec $\mathrm{PSL}_{2,q}(\mathbb{Z})$.

Proposition 1.3.5. *Soit* $\rho : \mathrm{PGL}_2(\mathbb{Z}) \longrightarrow \mathrm{PGL}_2(\Lambda)$ *une représentation, et supposons que* $\rho(T) = T_q$ *and* $\rho(S) = S_q$*. Alors* $\rho(N) = N_q$*.*

L'intérêt de cette extension réside dans son comportement vis-à-vis des deux versions de nombres déformés. Là où il y avait deux orbites sous l'action de $\mathrm{PSL}_2(\mathbb{Z})$, correspondant à la version droite et la version gauche, l'action de $\mathrm{PGL}_2(\mathbb{Z})$ unifie ces deux orbites en une seule.

Théorème 1.3.18 ([Jou25b])**.** *Soit* $M \in \mathrm{GL}_2(\mathbb{Z})$*, avec* $\det(M) = -1$*. Pour tout* $x \in \mathbb{P}^1(\mathbb{Q})$*,*

$$M_q \cdot [x]^\sharp = [M \cdot x]^\flat \text{ et } M_q \cdot [x]^\flat = [M \cdot x]^\sharp.$$

La deuxième extension que nous construisons nécessite l'opérateur de dualité entre q et $\frac{1}{q}$:

$$\begin{array}{rccc} \tau : & \mathbb{P}^1(\mathbb{Z}(q)) & \longrightarrow & \mathbb{P}^1(\mathbb{Z}(q)) \\ & f & \longrightarrow & f(q^{-1}) \end{array}.$$

Proposition 1.3.10 ([Jou25b]). *Il existe un unique élément $I_q \in \mathrm{PGL}_2(\Lambda)$ tel que l'opérateur $I_q \circ \tau$ commute avec l'action q-déformée de $\mathrm{PGL}_2(\mathbb{Z})$ sur $\mathbb{P}^1(\mathbb{Z}(q))$. Cet élément est donné par*

$$I_q = \begin{pmatrix} 1 & q-1 \\ 1-q & q \end{pmatrix},$$

et il satisfait $(I_q \circ \tau)^2 = 1$.

L'opérateur $I_q \circ \tau$ est interprété comme une deuxième déformation de l'identité, la première étant triviale. Cette transformation apparaît déjà dans le travail de Thomas [Tho24], où il introduit l'application de transition $g_q \mapsto \frac{1+(x-1)q}{1+(q-1)x}$. Le Théorème 1.2 de [Tho24] peut se comprendre grâce à la relation suivante

$$\tau \circ g_q = I_q \circ \tau = \overline{I_q}.$$

Pour toute matrice $M \in \mathrm{PGL}_2(\mathbb{Z})$, on définit similairement une deuxième déformation donnée par $\overline{M_q} = M_q \circ \overline{I_q}$. Cette deuxième déformation échange les versions droites et gauches en déterminant 1, et préserve les versions en déterminant -1.

Proposition 1.3.15 ([Jou25b]). *Soit $M \in \mathrm{PGL}_2(\mathbb{Z})$. Pour tout $x \in \mathbb{P}^1(\mathbb{Q})$, si $\det(M) = 1$,*

$$\overline{M}_q \cdot [x]^\sharp = [M \cdot x]^\flat \text{ et } \overline{M}_q \cdot [x]^\flat = [M \cdot x]^\sharp.$$

Inversement si $\det(M) = -1$,

$$\overline{M}_q \cdot [x]^\sharp = [M \cdot x]^\sharp \text{ et } \overline{M}_q \cdot [x]^\flat = [M \cdot x]^\flat.$$

Corollaire 1.3.16 ([Jou25b]). *Soit $x \in \mathbb{P}^1(\mathbb{Q})$. On a*

$$[-x]^\sharp = -q^{-1}[x]^\sharp_{q^{-1}} = \frac{-[x]^\flat_q + 1 - q^{-1}}{(q-1)[x]^\flat_q + 1},$$

$$\left[\frac{1}{x}\right]^\sharp = \frac{1}{[x]^\sharp_{q^{-1}}} = \frac{(q-1)[x]^\flat_q + 1}{q[x]^\flat_q + 1 - q}.$$

Nous étudions ensuite les aspects combinatoires des q-rationnels. Beaucoup de propriétés sont déjà connues, et ont fait l'objet de travaux antérieurs [MGO20, LMG21, LMGOV24, BBL23, Ren24] pour n'en citer que quelques-uns. Nous utilisons notre déformation des matrices de déterminant -1 pour apporter une contribution à ces travaux, en donnant une formule matricielle pour calculer les q-rationnels gauches. Cette formule utilise notamment la matrice suivante

$$J_q = N_q S_q = \begin{pmatrix} q-1 & 1 \\ q & 1-q \end{pmatrix}.$$

Définition 1.4.12. Soit x un nombre rationnel et $x = [a_1, a_2, \cdots, a_{2m+1}]$ son développement en fraction continue positive, de longueur impaire. On définit

$$M^{\mathrm{odd}}_{+,q}(x) = T_q^{a_1} J_q T_q^{a_2} \cdots T_q^{a_{2m+1}} J_q.$$

Proposition 1.4.13 ([Jou25b]). *Avec les mêmes notations, supposons que $m \geq 1$. Posons $x_{2m} = [a_1, \cdots, a_{2m}]$ et notons les déformations gauches de x et x_{2m}*

$$[x]^\flat = \frac{A^\flat}{B^\flat} \text{ et } [x_{2m}]^\flat = \frac{A^\flat_{2m}}{B^\flat_{2m}}.$$

Alors

$$M^{\text{odd}}_{+,q}(x) = q^{\min(0,a_1)} t_q^m \begin{pmatrix} qA^\flat & A^\flat_{2m} \\ qB^\flat & B^\flat_{2m} \end{pmatrix}.$$

Nous introduisons ensuite quatre déformations du déterminant de Farey entre une paire de nombres rationnels, et nous prouvons que ces q-déterminants sont des polynômes à coefficients positifs. Cela correspond à la propriété de positivité totale, l'un des fondements de la théorie de Morier-Genoud et Ovsienko [MGO20].

On fixe les notations suivantes. Pour une paire de nombres rationnels $(\frac{a}{b}, \frac{c}{d})$,

$$\left[\frac{a}{b}\right]^\sharp_q = \frac{A^\sharp}{B^\sharp}, \quad \left[\frac{a}{b}\right]^\flat_q = \frac{A^\flat}{B^\flat}, \quad \left[\frac{c}{d}\right]^\sharp_q = \frac{C^\sharp}{D^\sharp}, \quad \left[\frac{c}{d}\right]^\flat_q = \frac{C^\flat}{D^\flat}.$$

Définition 1.4.28 ([JPRT26]). Les *déterminants de Farey q-déformés* de la paire $(\frac{a}{b}, \frac{c}{d})$ sont

$$d^{\sharp\sharp}_F\left(\tfrac{a}{b}, \tfrac{c}{d}\right) = B^\sharp C^\sharp - A^\sharp D^\sharp \quad , \quad d^{\sharp\flat}_F\left(\tfrac{a}{b}, \tfrac{c}{d}\right) = B^\sharp C^\flat - A^\sharp D^\flat$$

$$d^{\flat\sharp}_F\left(\tfrac{a}{b}, \tfrac{c}{d}\right) = B^\flat C^\sharp - A^\flat D^\sharp \quad , \quad d^{\flat\flat}_F\left(\tfrac{a}{b}, \tfrac{c}{d}\right) = \frac{B^\flat C^\flat - A^\flat D^\flat}{q^2 - q + 1}.$$

Théorème 1.4.31 ([JPRT26]). *Les q-déterminants sont des polynômes en q à coefficients entiers, et pour toute paire de rationnels tels que $\frac{a}{b} < \frac{c}{d}$,*

$$B^\sharp C^\sharp - A^\sharp D^\sharp > 0 \quad , B^\sharp C^\flat - A^\sharp D^\flat > 0$$
$$B^\flat C^\sharp - A^\flat D^\sharp > 0 \quad , \frac{B^\flat C^\flat - A^\flat D^\flat}{q^2 - q + 1} > 0$$

où on note $P > 0$ pour un polynôme P dont les coefficients sont positifs.

Enfin, nous concluons le chapitre par une étude de certains nombres algébriques, en utilisant l'action étendue de $\mathrm{PGL}_2(\mathbb{Z})$ sur les q-réels. De manière générale, un nombre réel déformé $[x]_q$ est défini comme une série de Laurent en q, limite des déformations de toute suite de rationnels convergeant vers x.

Proposition 1.5.1 ([MGO22]). *Soit $x \in \mathbb{R} \setminus \mathbb{Q}$. Pour toute suite de rationnels $(x_n)_n$ convergeant vers x, les développements en séries de Taylor de $[x_n]^\sharp_q$ convergent, au sens des séries de Laurent, vers une série qui ne dépend pas du choix de la suite de rationnels $(x_n)_n$.*

Définition 1.5.2 ([MGO22]). Avec les notations précédentes, la série limite des $[x_n]^\sharp_q$ est notée $[x]_q$, et constitue la déformation du nombre x.

On peut montrer que le résultat est le même si on utilise la version gauche pour la suite de rationnels.

Proposition 1.5.3 ([Jou25b]). *Soit $x \in \mathbb{R} \setminus \mathbb{Q}$. Soit $(x_n)_n$ une suite de nombres rationnels convergeant vers x. Alors la suite $([x_n]^\flat)$ converge vers $[x]_q$ au sens des séries de Laurent.*

L'action déformée du groupe modulaire étendu $\mathrm{PGL}_2(\mathbb{Z})$ sur les q-rationnels passe à la limite, et on obtient donc une action de $\mathrm{PGL}_2(\mathbb{Z})$ sur les q-irrationnels.

Proposition 1.5.5 ([LMG21, Jou25b]). *Soit $x \in \mathbb{R} \setminus \mathbb{Q}$. Soit $M \in \mathrm{PGL}_2(\mathbb{Z})$. Alors*

$$M_q \cdot [x]_q = [M \cdot x]_q.$$

En utilisant l'équivariance des q-irrationnels sous l'action de $\mathrm{PSL}_2(\mathbb{Z})$, Leclere et Morier-Genoud ont pu donner des formules closes pour les déformations de nombres quadratiques [LMG21]. Nous étudions le cas de certains nombres algébriques de degrés 4 et 6.

Considérons le polynôme $x^4 - bx^2 + 1$, dépendant d'un paramètre entier b, et notons X_1, X_2, X_3 et X_4 les q-déformations de ses racines. On suppose que $b > 2$.

Proposition 1.5.16 ([Jou25b]). *Les q-nombres X_1, X_2, X_3 et X_4 sont solutions de l'équation déformée suivante*

$$X^4 - \Sigma_1 X^3 + \left(\frac{q + q^{-1} - 3}{q - 1} \Sigma_1 + 2q^{-1} \right) X^2 + q^{-1} \Sigma_1 X + q^{-2} = 0,$$

où $\Sigma_1 = X_1 + X_2 + X_3 + X_4$ est une série de Laurent en q.

Considérons à présent le polynôme $x^6 - 3x^5 - bx^4 + (2b + 5)x^3 - bx^2 - 3x + 1$. Comme avant, on note X_i les déformations de ses racines. On suppose que $b \geq 1$.
On note Σ_i, pour $i \in [\![1, 6]\!]$, les polynômes symmétriques en les X_i.

Proposition 1.5.21 ([Jou25b]). *Les q-nombres X_i vérifient les relations coefficients-racines déformées suivantes*

- $\Sigma_6 = 1$,
- $\Sigma_5 = -\Sigma_1 + 6$,
- $\Sigma_4 = -5\Sigma_1 + \Sigma_2 + 15$,
- $\Sigma_3 = -5\Sigma_1 + 2\Sigma_2 + 10$,
- $(q - 1)\Sigma_2 = (q^2 + q - 4 + q^{-1})\Sigma_1 + 3(-q^2 + q + 2 - q^{-1})$.

Abrégé du Chapitre 2 : Traces et q-nombres de Markov

Les nombres de Markov forment une famille d'entiers positifs définis comme solutions d'une équation Diophantienne appelée équation de Markov. Ils jouent un rôle important en théorie des nombres notamment grâce à un théorème célèbre d'Andrey Markov, qui met en lumière comment ces nombres apparaissent naturellement dans la théorie de l'approximation Diophantienne [Mar79]. En plus de ce résultat majeur, les nombres de Markov sont également intéressants dans d'autres champs mathématiques : en combinatoire, en géométrie hyperbolique ou en physique mathématique.

Lorsque les nombres rationnels q-déformés de Morier-Genoud et Ovsienko ont vu le jour en 2020, Kogiso a appliqué cette construction aux nombres de Markov et ainsi défini une notion de q-nombre de Markov [Kog20], notion qui est l'objet de ce chapitre. Peu de temps après, Labbé et Lapointe ont étudié la combinatoire d'une version différente (mais très proche) des q-nombres de Markov [LL22]. Un article itérant sur cette version a été publié quelques années après par Labbé, Lapointe et Steiner [LLS24], et très récemment Aval et Labbé ont donné une interprétation combinatoire de cette version en termes d'appariements parfaits dans des graphes en serpents. Du côté de la version des q-nombres de Markov définis par Kogiso, une interprétation combinatoire a aussi été trouvée, utilisant des posets zigzags circulaires, par Kantarcı Oğuz [KO25]. Avec Sam Evans, Sophie Morier-Genoud et Valentin Ovsienko, nous avons contribué à ce sujet en prouvant des résultats d'unicité et en donnant une autre interprétation combinatoire des q-nombres de Markov, fondée sur des appariements parfaits avec poids [EJMGO26]. Par ailleurs, à la fin du Chapitre 3, nous allons décrire des analogues des nombres de Markov encore légèrement différents des deux précédentes versions, également définis à partir de la théorie des q-rationnels.

Dans ce chapitre, nous nous intéressons donc à une déformation des nombres de Markov construite à partir des traces de certaines q-matrices que nous avons introduites au Chapitre 1. Nous commençons par une étude générale des traces de q-matrices.

Traces des q-matrices

Définition 2.1.1. Soit $f(q) \in \mathbb{Z}[q]$. On dit que le polynôme $f(q)$ est palindromique lorsque la suite de ses coefficients l'est, i.e. si $q^{\deg(f)} f(q^{-1}) = f(q)$.

Théorème 2.1.2 ([LMG21, Jou25b])**.** *Soit $M \in \mathrm{PSL}_2(\mathbb{Z})$. Alors $\mathrm{Tr}(M_q)$ est un polynôme palindromique à coefficients entiers positifs, à signe près et à une puissance de q près.*

Dans son article [KO25], Kantarcı Oğuz démontre que les traces des q-matrices de $\mathrm{PSL}_{2,q}(\mathbb{Z})$ sont des fonctions génératrices de posets zigzags circulaires. Nous étendons ce résultat aux matrices de déterminant -1, dont les q-analogues ont été définis au chapitre précédent. Pour cela, nous introduisons la notion de poset zigzag circulaire gauche associé à un nombre rationnel, et la notion d'idéal admissible dans un tel poset.

Proposition 2.1.16 ([Jou25b])**.** *Soit $x \in \mathbb{Q}_{\geq 1}$ et soit $M^{\mathrm{odd}}_{+,q}(x) \in \mathrm{PGL}_{2,q}(\mathbb{Z})$ la matrice de déterminant -1 associée à x. Alors à puissances de $t_q = 1 - q + q^2$ près, la trace de cette matrice est la fonction génératrice des idéaux admissibles du poset zigzag circulaire gauche associé à x.*

Construction classique des nombres de Markov

Avant de s'intéresser à leurs déformations, nous présentons la construction classique des nombres de Markov, telle que présentée dans le livre d'Aigner [Aig13]. L'équation de Markov est la suivante

$$x^2 + y^2 + z^2 = 3xyz.$$

Un triplet de Markov est un triplet d'entiers positifs (a, b, c), solution de l'équation de Markov. Un nombre de Markov est un entier apparaissant dans un triplet de Markov. Les triplets de Markov peuvent être tous calculés à partir du triplet initial $(1, 1, 1)$, en effectuant des transformations successives appelées mutations. Ce processus permet de ranger tous les triplets de Markov dans un arbre binaire appelé l'arbre de Markov.

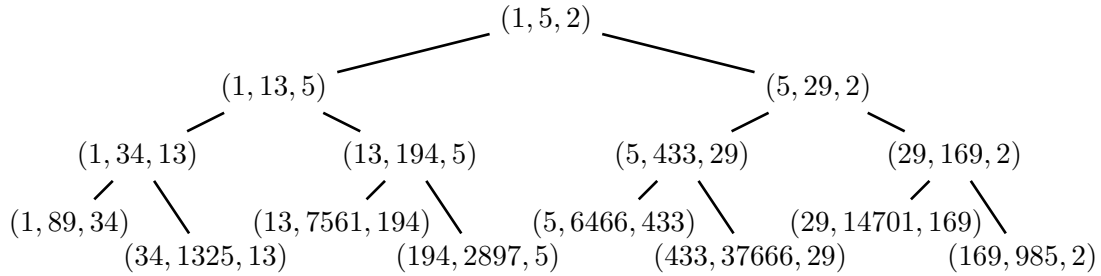


Figure 2.4: Premiers étages de l'arbre de Markov

Une conjecture devenue célèbre par sa difficulté, due à Frobenius [Fro13], stipule que chaque nombre de Markov apparaît une unique fois comme maximum d'un triplet dans l'arbre de Markov. Des reformulations de cette conjecture ont été établies, en utilisant notamment des passerelles entre l'arbre de Markov et d'autres arbres célèbres en combinatoire. Nous résumons ces passerelles dans le diagramme suivant.

$$\begin{array}{ccc} \text{Arbre de Farey} & & \text{Mots de Christoffel} \\ (s,t,u) & \longleftrightarrow & (w^s, w^t, w^u) \\ t = s \oplus_F u & & w^t = w^s w^u \\ \updownarrow & & \updownarrow \\ \text{Matrices de Cohn} & & \text{Triplets de Markov} \\ (C^s, C^t, C^u) & \longleftrightarrow & (m^s, m^t, m^u) \\ C^t = C^s C^u & & (m^t)^2 + (m^s)^2 + (m^u)^2 = 3m^s m^t m^u. \end{array}$$

En particulier, les matrices de Cohn sont des matrices de $\mathrm{SL}_2(\mathbb{Z})$ dont l'entrée en haut à droite est un nombre de Markov, qui coïncide avec le tiers de la trace de la matrice. Les matrices de Cohn sont classifiées en familles infinies, paramétrées par les entiers relatifs. Pour chaque entier $n \in \mathbb{Z}$, les $n$ièmes matrices de Cohn sont des produits des deux matrices élémentaires suivantes :

$$A(n) = \begin{pmatrix} n & 1 \\ 3n - n^2 - 1 & 3 - n \end{pmatrix} \text{ and } B(n) = \begin{pmatrix} 2n+1 & 2 \\ 4n - 2n^2 + 2 & 5 - 2n \end{pmatrix}.$$

Grâce à la correspondance entre l'arbre de Farey et l'arbre de Markov, les matrices de Cohn d'une même famille sont indexées par les nombres rationnels. Ainsi, pour tout $t \in (0, 1)$, on associe une matrice de Cohn $C^t(n)$ dans la $n$ième famille.

Une déformation des nombres de Markov

Comme nous l'avons vu, les traces des q-matrices ont des propriétés combinatoires particulièrement intéressantes. Il semble donc naturel de définir des analogues de nombres de Markov en prenant la trace des q-versions des matrices de Cohn dans $\mathrm{PSL}_{2,q}(\mathbb{Z})$. Cette idée a été explorée par Kogiso [Kog20] et Kantarcı Oğuz [KO25] dans le cas d'une des familles de matrice de Cohn, la plus utilisée. Nous avons repris cette idée en l'appliquant à toutes les matrices de Cohn, dans le cadre d'une collaboration avec Sam Evans, Sophie Morier-Genoud et Valentin Ovsienko [EJMGO26].

Les analogues des matrices de Cohn élémentaires $A(n)$ et $B(n)$ sont les suivantes.

$$A(n)_q = \begin{pmatrix} q^{2-n}[n]_q & q^{1-n} \\ [n]_q[3-n]_q - q^{n-1} & q^{-1}[3-n]_q \end{pmatrix}.$$

$$B(n)_q = \begin{pmatrix} q^{1-n}[n+1]_q[2]_q - q & q^{-n}[2]_q \\ [2]_q\big((q+q^{-1})[n]_q - q^{2-n}[n+1]_q[n-1]_q\big) & q^{-2}[3-n]_q[2]_q - q^{-1} \end{pmatrix}.$$

L'un de nos résultats principaux est l'unicité des q-nombres de Markov définis par cette technique. Considérons la déformation suivante de l'équation de Markov :

$$x^2 + y^2 + z^2 \;=\; q^{-1}\,[3]_q\,xyz \;+\; (q-1)\,(q^{-1}-1). \tag{2.14}$$

Théorème 2.3.10 ([EJMGO26])**.** *Pour tout triplet de Markov (a,b,c), il existe un unique triplet de polynômes de Laurent à coefficients entiers (a_q, b_q, c_q) solution de l'équation de Markov déformée, et qui vaut (a,b,c) en $q=1$. Ces polynômes sont donnés par la trace des q-matrices de Cohn divisée par $[3]_q$, indépendamment du paramètre n.*

Le deuxième théorème important de ce travail fournit une interprétation combinatoire des q-nombres de Markov, et permet de déduire un analogue de la fameuse conjecture de Frobenius. Pour cela, nous munissons de poids les arêtes des graphes en serpent associés aux nombres de Markov. Ces graphes lestés sont notés $\mathcal{G}(q)$.

Théorème 2.3.24 ([EJMGO26])**.** *Soit m un nombre de Markov et m_q son analogue. Soit $\mathcal{G}(q)$ un graphe lesté associé à m. Alors m_q est la fonction génératrice des appariements parfaits lestés dans $\mathcal{G}(q)$.*

Corollaire 2.3.26 ([EJMGO26])**.** *Chaque q-nombre de Markov apparaît une unique fois comme polynôme de plus haut degré d'un triplet dans l'arbre de Markov déformé.*

Abrégé du Chapitre 3 : Aspects géométriques des q-nombres

La force des q-nombres réels est qu'ils croisent de nombreux champs des mathématiques et peuvent être compris dans plusieurs contextes. Dans les chapitres précédents, notre approche était principalement combinatoire, et nous explorons à présent une facette géométrique des q-nombres, en travaillant dans le plan hyperbolique. Les résultats de ce chapitre ont été développés en collaboration avec Olga Paris-Romaskevich et Alexander Thomas [JPRT26].

Généralisation des opérations de Farey

Les q-nombres rationnels peuvent être calculés au moyen d'une déformation du graphe de Farey [MGO20]. En particulier, toute paire de rationnels $(\frac{a}{b}, \frac{c}{d})$ telle que $|ad-bc|=1$ (la paire est à distance 1 dans le graphe de Farey) donne lieu à une déformation de la somme de Farey classique :

$$\left[\frac{a+c}{b+d}\right]_q^\sharp = \frac{A^\sharp + q^\alpha C^\sharp}{B^\sharp + q^\alpha D^\sharp}, \text{ et } \left[\frac{a+c}{b+d}\right]_q^\flat = \frac{q^\beta A^\flat + C^\flat}{q^\beta B^\flat + D^\flat}.$$

Nous étendons ces formules à des paires de rationnels dont la distance dans le graphe de Farey est égale à 2. Pour des paires plus éloignées, la question devient beaucoup plus difficile, car alors la forme réduite de la somme de Farey n'est plus contrôlée.

Théorème 3.2.7 ([JPRT26]). *Soit $(\frac{a}{b}, \frac{c}{d})$ une paire de rationnels. On note $d_F = |ad-bc|$ le déterminant de Farey. Alors cette paire est à distance 1 ou 2 dans le graphe de Farey si et seulement si $\gcd(a+c, b+d) = d_F$ ou $\gcd(a-c, b-d) = d_F$.*
De plus, si $\gcd(a+c, b+d) = d_F$, alors il existe deux entiers α, β et une déformation du déterminant de Farey $d_F^{\sharp\sharp}$ tels que

$$\left[\frac{a+c}{b+d}\right]_q^\sharp = \frac{(q^\alpha A^\sharp + q^\beta C^\sharp)/d_F^{\sharp\sharp}}{(q^\alpha B^\sharp + q^\beta D^\sharp)/d_F^{\sharp\sharp}},$$

où la fraction du membre de droite est réduite.
De même, si $\gcd(a-c, b-d) = d_F$, alors il existe deux entiers α, β tels que

$$\left[\frac{a}{b} \ominus_F \frac{c}{d}\right]_q^\sharp = \frac{(q^\alpha A^\sharp - q^\beta C^\sharp)/d_F^{\sharp\sharp}}{(q^\alpha B^\sharp - q^\beta D^\sharp)/d_F^{\sharp\sharp}},$$

où la fraction du membre de droite est réduite.

Les q-nombres sous forme de disques

Nous allons associer à chaque nombre réel un disque dans le plan, le long de l'axe réel, dont le rayon et la position sont donnés par son analogue quantique. Cette idée était déjà en germe dans l'article de Bapat, Becker et Licata [BBL23], et nous en explorons ici les implications.

Définition 3.3.1 ([JPRT26]). Fixons $q \in (0,1)$. Soit $x \in \mathbb{P}^1(\mathbb{Q})$. Le disque associé à x, noté $[x]$, est l'unique disque orthogonal à l'axe réel et le coupant en $[x]_q^\flat$ et $[x]_q^\sharp$.
L'union de tous les disques $[x]$, pour $x \in \mathbb{P}^1(\mathbb{Q})$, est notée $\mathcal{Q}$.

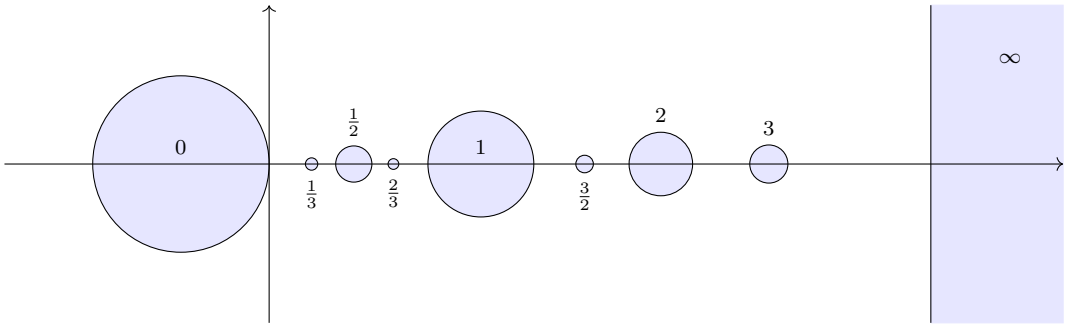


Figure 3.5: Les disques associés aux nombres rationnels de profondeur au plus 3 dans le graphe de Farey. Le paramètre q est fixé à 0.6.

Remarquons que comme la déformation préserve l'ordre des rationnels (propriété de positivité totale), les disques que nous avons défini sont tous disjoints. Cela peut paraître contre-intuitif, mais cela est permis par le fait que pour $x \in \mathbb{Q}$, l'intervalle $[[x]_q^\flat, [x]_q^\sharp]$ ne contient pas x en général.

Opérations de Springborn

En prenant les tangentes communes à deux cercles, on peut définir deux nouveaux points, appelés centres homothétiques intérieur et extérieur. Nous appliquons cette opération aux disques associés aux q-rationnels, et faisons le lien avec des opérations étudiées par Springborn.

Pour deux cercles C_1, et C_2, on note $i(C_1, C_2)$ le centre homothétique intérieur et $e(C_1, C_2)$ le centre homothétique extérieur.

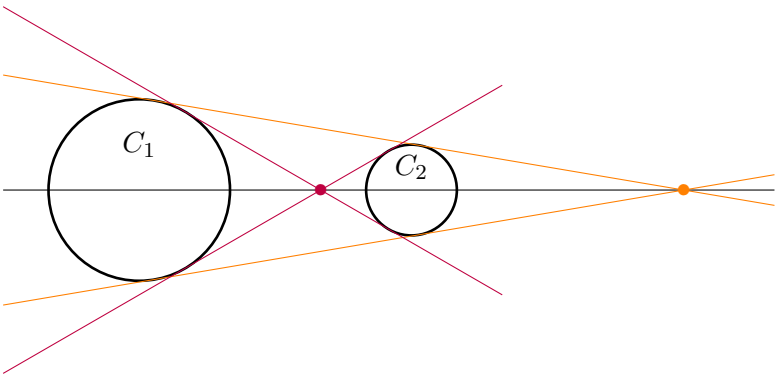


Figure 3.6: Les centres homothétiques de deux cercles, intérieur en rose et extérieur en orange.

Expérimentalement, nous observons que pour de nombreuses paires de rationnels $\left(\frac{a}{b}, \frac{c}{d}\right)$, les centres homothétiques intérieurs et extérieurs des disques associés sont au bord d'autres q-disques, dont les étiquettes sont données par

$$i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \left[\frac{ab+cd}{b^2+d^2}\right]_q^\sharp \text{ and } e\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \left[\frac{ab-cd}{b^2-d^2}\right]_q^\flat.$$

Afin de rendre précise cette observation, nous définissons une notion de paire régulière.

Définition 3.4.6 ([JPRT26]). Une paire $(\frac{a}{b}, \frac{c}{d}) \in \mathbb{Q}^2$ dont le déterminant de Farey est $d_F = |ad - bc|$ est dite *régulière à l'intérieur* lorsque

$$\gcd(ab + cd, b^2 + d^2, a^2 + c^2) = d_F.$$

De même, la paire est dite *régulière à l'extérieur* lorsque $\gcd(ab - cd, b^2 - d^2, a^2 - c^2) = d_F$.

Pour ces paires, la forme réduite des opérations de Springborn déformées est explicite.

Théorème 3.5.5 ([JPRT26]). *Soit $(\frac{a}{b}, \frac{c}{d}) \in \mathbb{Q}^2$ une paire régulière à l'intérieur. Alors il existe des entiers ε_1 et ε_2 et une déformation du déterminant de Farey $d_F^{\sharp\flat}$ tels que, à puissance de q près,*

$$\gcd(q^{\varepsilon_2} A^\sharp B^\flat + q^{\varepsilon_1} C^\flat D^\sharp, q^{\varepsilon_2} B^\sharp B^\flat + q^{\varepsilon_1} D^\flat D^\sharp, q^{\varepsilon_2} A^\sharp A^\flat + q^{\varepsilon_1} C^\flat C^\sharp) = d_F^{\sharp\flat}.$$

De même si la paire est régulière à l'extérieur, on a

$$\gcd(q^{\varepsilon_2} A^\sharp B^\flat - q^{\varepsilon_1} C^\sharp D^\flat, q^{\varepsilon_2} B^\sharp B^\flat - q^{\varepsilon_1} D^\flat D^\sharp, q^{\varepsilon_2} A^\sharp A^\flat - q^{\varepsilon_1} C^\flat C^\sharp) = d_F^{\sharp\sharp}.$$

Enfin, on caractérise les paires régulières et on démontre l'observation sur le lien avec les centres homothétiques.

Théorème 3.5.9 ([JPRT26]). *Soit $(\frac{a}{b}, \frac{c}{d}) \in \mathbb{Q}^2$ et notons d_F le Farey déterminant de cette paire. Les conditions suivantes sont équivalentes :*

(1) la paire est régulière à l'extérieur,

(2) $\gcd(a + c, b + d) \gcd(a - c, b - d) \in \{d_F, 2d_F\}$,

(3) $d_F\left(\frac{a+c}{b+d}, \frac{a-c}{b-d}\right) \in \{1, 2\}$,

(4) il y a une involution renversant l'orientation dans $\mathrm{PGL}_2(\mathbb{Z})$ *qui échange $\frac{a}{b}$ et $\frac{c}{d}$.*

De même pour la régularité à l'extérieur, en travaillant dans $\mathbb{Q}[i]$:

(1') la paire est régulière à l'extérieur,

(2') $\gcd(a + ic, b + id) \gcd(a - ic, b - id) = d_F$,

(3') $d_F\left(\frac{a+ic}{b+id}, \frac{a-ic}{b-id}\right) = 2$,

(4') il y a une involution préservant l'orientation dans $\mathrm{PGL}_2(\mathbb{Z})$ *qui échange $\frac{a}{b}$ et $\frac{c}{d}$.*

Théorème 3.5.10 ([JPRT26]). *Soit $(\frac{a}{b}, \frac{c}{d}) \in \mathbb{Q}^2$. Si la paire est régulière à l'intérieur, alors*

$$\left[\frac{a}{b} \oplus_S \frac{c}{d}\right]_q^\sharp = i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right).$$

Si la paire est régulière à l'extérieur, alors

$$\left[\frac{a}{b} \ominus_S \frac{c}{d}\right]_q^\flat = e\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right).$$

Application aux fractions de Markov

Les fractions de Markov ont été définies par Springborn [Spr24], et forment une famille de nombres rationnels dont les dénominateurs sont les nombres de Markov. Elles fournissent une classe d'exemples de paires régulières à l'intérieur.

On peut définir les fractions de Markov en itérant la somme de Springborn à partir de la paire de départ $(\frac{0}{1}, \frac{1}{2})$.

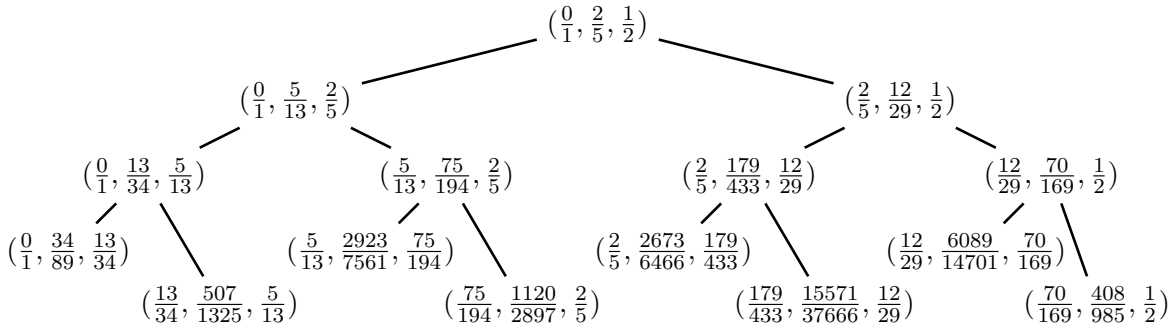


Figure 3.7: Premiers étages de l'arbre des fractions de Markov

En déformant ces fractions de Markov, on aboutit à une nouvelle notion de q-nombre de Markov.

Théorème 3.6.3 ([JPRT26]). *Si* $\left(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$ *est un triplet de fractions de Markov, notons* $\frac{A_i^\sharp}{B_i^\sharp}$ *les déformations droites et* $\frac{A_i^\flat}{B_i^\flat}$ *les déformations gauches, pour* $i \in \{0,1,2\}$. *Alors*

$$\begin{cases} B_1^\sharp B_1^\flat + q^{\varepsilon_0+3} B_2^\sharp B_2^\flat + B_0^\flat (B_1^\sharp A_2^\sharp - q^3 A_1^\sharp B_2^\sharp) = [3]_q B_1^\sharp B_2^\sharp B_0^\flat & (r_0) \\ B_0^\sharp \equiv B_1^\sharp A_2^\sharp - A_1^\sharp B_2^\sharp & (r_1) \\ B_0^\flat \equiv B_1^\sharp A_2^\flat - A_1^\sharp B_2^\flat & (r_1^\flat) \\ B_2^\sharp \equiv A_1^\sharp B_0^\sharp - B_1^\sharp A_0^\sharp & (r_2) \\ B_2^\flat \equiv A_1^\flat B_0^\sharp - B_1^\flat A_0^\sharp & (r_2^\flat) \end{cases} \tag{3.15}$$

où ε_0 *est un entier associé à* $\frac{a_0}{b_0}$.

La preuve de ce résultat repose sur une interprétation combinatoire des fractions de Markov déformées, à l'aide de posets zigzags, ainsi que sur l'application du Théorème 3.5.10.

Abrégé du Chapitre 4 : Tresses et q-rationnels en dimension 2

Comme nous l'avons vu, la théorie des q-rationnels repose de manière cruciale sur la déformation de l'action du groupe modulaire sur la droite rationnelle. Comme l'ont montré Morier-Genoud, Ovsienko et Veselov [MGOV24], cette déformation coïncide en fait avec la représentation de Burau du groupe de tresse B_3, via l'isomorphisme entre le groupe modulaire et B_3 quotienté par son centre. Comme B_3 fait partie d'une famille de groupes de tresses définis pour tout entier $n \geq 3$, il est tentant d'essayer de généraliser la construction des q-rationnels en plus grande dimension grâce aux représentations de Burau des groupes de tresses supérieurs. Nous étudions ici le cas du groupe B_4 agissant sur le plan.

Représentation de Burau entière

Nous commençons par étudier la spécialisation en $q = 1$ de la représentation de Burau de B_4 (avec les notations usuelles de la littérature [KT08], cela correspond à $t = -1$).

Définition 4.1.14. La représentation de Burau entière de B_4 est le morphisme de groupes suivant,

$$\rho_4^i : B_4 \to \mathrm{PSL}(3, \mathbb{Z}),$$

défini sur les générateurs usuels de B_4 par

$$\rho_4^i(\sigma_1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_4^i(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_4^i(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

L'action $B_4 \curvearrowright \mathbb{P}^2(\mathbb{Q})$ est engendrée par

$$\begin{aligned} \rho_4^i(\sigma_1) : \quad [r : s : t] &\mapsto [r + s : s : t], \\ \rho_4^i(\sigma_2) : \quad [r : s : t] &\mapsto [r : s + t - r : t], \\ \rho_4^i(\sigma_3) : \quad [r : s : t] &\mapsto [r : s : t - s]. \end{aligned}$$

Le premier résultat important de ce chapitre classifie les orbites de l'action ci-dessus.

Théorème 4.2.5 ([Jou25a])**.** *(i) Sous l'action de B_4, le plan projectif rationnel est décomposé en une infinité d'orbites, comme suit*

$$\mathbb{P}^2(\mathbb{Q}) = \{[1 : 0 : 1]\} \sqcup \mathrm{Orb}_{B_4}([0 : 1 : 0]) \sqcup \bigsqcup_{\substack{n \geq 2, 0 < m < n/2 \\ m \wedge n = 1}} \mathrm{Orb}_{B_4}([m : n : m]).$$

(ii) Pour tout couple d'entiers premiers entre eux (m, n), l'orbite $\mathrm{Orb}_{B_4}([m : n : m])$ est décrite par

$$\mathrm{Orb}_{B_4}([m : n : m]) = \left\{ [r : s : t] \in \mathbb{P}^2(\mathbb{Q}) \;\middle|\; \begin{cases} \gcd(r - t, s) = n; \\ r, t \equiv \pm m \mod (n). \end{cases} \right\},$$

et

$$\mathrm{Orb}_{B_4}([0 : 1 : 0]) = \{[r : s : t] \mid \gcd(r - t, s) = 1\}.$$

Ce résultat repose en partie sur l'élaboration d'un algorithme explicite pour relier un point du plan à un représentant de son orbite. Nous appelons cette procédure l'algorithme d'Euclide tressé.

Algorithme d'Euclide tressé

Entrée: Commencer avec un point $[r : s : t] \neq [1 : 0 : 1]$, pour $r, s, t \in \mathbb{Z}$, premiers entre eux dans leur ensemble. Supposer que $s \geq 0$.

Etape 1: Si $s = 0$, appliquer σ_2 pour remplacer s par $t - r$, et si nécessaire changer les signes pour conserver $s > 0$.
Ecrire la division euclidienne de r par s et de t par s :

$$r = sa_1 + r' \qquad t = sc_1 + t'$$

Appliquer $\sigma_1^{-a_1}\sigma_3^{c_1}$, pour avoir $[r : s : t] \mapsto [r' : s : t'] =: [r_1 : s_1 : t_1]$.

Etape 2: Tant que $r_i - t_i \neq 0$, répéter :
Ecrire la division euclidienne supérieure de s_i par $(r_i - t_i)$, i.e.

$$s_i = (r_i - t_i)b_{2i} + s' \text{ with } 0 < s' \leq |r_i - t_i| .$$

Appliquer $\sigma_2^{b_{2i}}$ et poser $s_{i+1} := s'$.
Ecrire les divisions euclidiennes de r_i et t_i par s_{i+1} :

$$r_i = s_{i+1}a_{2i+1} + r' \qquad t_i = s_{i+1}c_{2i+1} + t'.$$

Appliquer $\sigma_1^{-a_{2i+1}}\sigma_3^{c_{2i+1}}$ et poser $r_{i+1} := r'$ et $t_{i+1} := t'$.
Terminaison: Si $r_i = t_i$, terminer.

En complément, nous donnons également une description explicite des stabilisateurs de chaque orbite, modulo le noyau de la représentation de Burau entière, noté $\mathcal{BI}_4$.

Notation 4.2.9. Dans B_4, on considère $\tau_1 = (\sigma_1\sigma_2\sigma_1)^2$, $\tau_3 = (\sigma_3\sigma_2\sigma_3)^2$, et $\Delta_4 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$.

Théorème 4.2.10 ([Jou25a])**.** *Soit $n \in \mathbb{N}^*$ et $0 \leq m < n$ premier avec n. Alors*

$$\mathrm{Stab}_{[m:n:m]}/\mathcal{BI}_4 = \begin{cases} \langle \tau_1\Delta_4,\ \sigma_2 \rangle & \text{si } n \geq 3, \\ \langle \tau_1\Delta_4,\ \sigma_2,\ \sigma_1\sigma_2^2\sigma_3 \rangle & \text{si } n = 2, \\ \langle \tau_1,\ \Delta_4,\ \sigma_2 \rangle & \text{si } n = 1. \end{cases}$$

Nous nous intéressons ensuite à la répartition des orbites dans le plan, esquissant expérimentalement leur structure, puis démontrant cette structure.

Théorème 4.2.12 ([Jou25a])**.**
1. *Toutes les orbites excepté le singleton $\{[1 : 0 : 1]\}$ sont denses dans le plan.*
2. *L'orbite de $[0 : 1 : 0]$ est la seule contenant une infinité de droites affines de $\mathbb{Q}^2$.*
3. *Les orbites sont stables sous l'action du groupe dihédral D_4, lorsque l'on place le centre de l'action en $[1 : 0 : 1]$.*

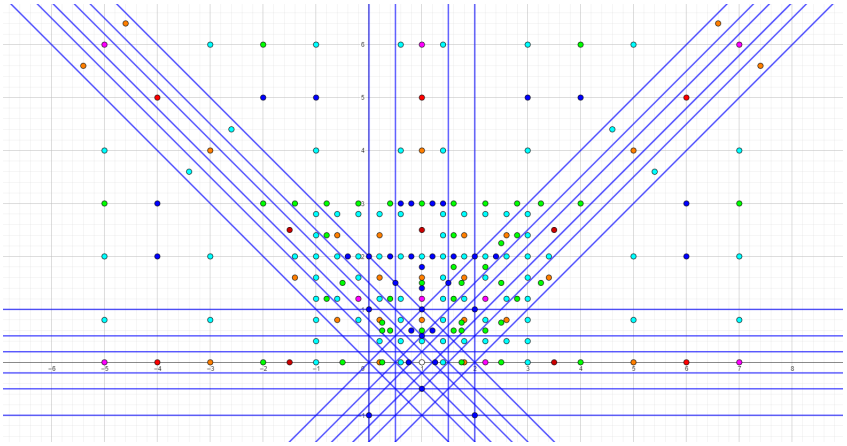


Figure 4.8: Esquisse des orbites dans $\mathbb{P}^2(\mathbb{Q})$, représenté dans la carte affine $[x : y : 1]$. Chaque couleur réprésente une orbite, le bleu foncé étant pour l'orbite de $[0 : 1 : 0]$.

Déformation du plan

À présent que la situation classique, en $q = 1$, est bien comprise, nous nous intéressons au cas déformé. En particulier, les résultats précédents nous motivent à nous concentrer sur l'orbite de $[0 : 1 : 0]$, que nous appelons l'orbite principale, notée $\mathcal{O}_1$. La représentation de Burau est donnée sur les générateurs de B_4 par

$$\rho_4 : B_4 \to \mathrm{GL}_3(\mathbb{Z}[q, q^{-1}]),$$

$$\rho_4(\sigma_1) = \begin{pmatrix} q & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_4(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ -q & q & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_4(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & q \end{pmatrix}.$$

Cela induit une action $B_4 \curvearrowright \mathbb{P}^2(\mathbb{Z}(q))$, décrite par

$$\begin{aligned} \rho_4(\sigma_1) : \quad [r : s : t] &\mapsto [qr + s : s : t], \\ \rho_4(\sigma_2) : \quad [r : s : t] &\mapsto [r : qs + t - qr : t], \\ \rho_4(\sigma_3) : \quad [r : s : t] &\mapsto [r : s : qt - qs]. \end{aligned}$$

Définition 4.3.15 ([Jou25a])**.** Par analogie avec le cas $q = 1$, notons $\mathcal{O}_q$ l'orbite sous l'action de B_4 du point $[0 : 1 : 0] \in \mathbb{P}^2(\mathbb{Z}(q))$. Nous l'appelons l'orbite principale q-déformée.

Définition 4.3.3. L'application de déformation est une fonction multi-valuée définie par

$$\begin{array}{rccc} \mathcal{Q} : & \mathcal{O}_1 & \longrightarrow & \mathcal{P}(\mathbb{P}^2(\mathbb{Z}(q))) \\ & p & \longmapsto & \{\rho_4(\beta)([0 : 1 : 0]) \mid \beta \text{ s.t. } \rho_4^i(\beta)([0 : 1 : 0]) = p\} \end{array}.$$

Pour tout $p \in \mathcal{O}_1$, l'image $\mathcal{Q}(p)$ est appelée la quantification de p, et un élément $[R : S : T]$ de $\mathcal{Q}(p)$ sera appelé une déformation de p.

Le principal intérêt de cette déformation est sa compatibilité avec la théorie des q - rationnels. Considérons le plongement suivant de la droite dans le plan.

$$\mathbb{P}^1(\mathbb{Q}) \overset{\iota}{\hookrightarrow} \mathbb{P}^2(\mathbb{Q}), \qquad [r : s] \mapsto [r : s : 0].$$

De même dans le contexte quantifié :

$$\mathbb{P}^1(\mathbb{Z}(q)) \overset{\iota_q}{\hookrightarrow} \mathbb{P}^2(\mathbb{Z}(q)), \qquad [R(q) : S(q)] \mapsto [R(q) : S(q) : 0].$$

Théorème 4.3.6 ([Jou25a])**.** *La quantification de la droite projective au sens de [MGO20] et notre quantification du plan commutent avec les plongements de la droite dans le plan. En d'autres termes, pour tout* $[r : s] \in \mathbb{P}^1(\mathbb{Q})$*,*

$$\iota_q(\mathcal{Q}([r : s])) \in \mathcal{Q}(\iota([r : s])).$$

Afin de résoudre l'indétermination qui plane entre les différentes déformations possibles pour un point du plan, nous émettons la conjecture d'unicité suivante, étayée par de nombreuses expérimentations informatiques.

Conjecture 4.3.16. *Soit* $p \in \mathcal{O}_1$*. Il y a une unique* $[R : S : T]$ *de* p *dans* $\mathcal{Q}(p)$ *telle que* $\deg(R)$*,* $\deg(S)$ *et* $\deg(T)$ *sont minimaux.*

Définition 4.3.17. En admettant la conjecture ci-dessus, on peut définir la déformation d'un point $p \in \mathcal{O}_1$ comme la déformation minimale (en degrés) de p. Si $p = [r : s : t]$, on note cette déformation minimale $[r : s : t]_q$.

Cas des groupes de tresses supérieurs

En vue de potentiels futurs travaux, nous étudions l'action de B_{n+2} sur $\mathbb{P}^n(\mathbb{Q})$ pour tout $n \geq 1$, définie par la représentation de Burau entière. Sur les générateurs standards, cette action est

$$\sigma_i \cdot [x_1 : \cdots : x_{i-1} : x_i : x_{i+1} \cdots : x_{n+1}] = [x_1 : \cdots : x_{i-1} : x_i + x_{i+1} - x_{i-1} : x_{i+1} : \cdots : x_{n+1}]$$

Comme le laissaient supposer les cas $n = 3$ et $n = 4$, la situation change en fonction de la parité de n.

Théorème 4.4.3. *Soit* $n = 2m + 1$ *un entier impair. Alors l'action* $B_{n+2} \curvearrowright \mathbb{P}^n(\mathbb{Q})$ *a* $m + 1$ *orbites, et un système de représentants est donné par*

$$\{[1 : 0 : 0 : \varepsilon_1 : 0 : \varepsilon_2 : 0 : \cdots : \varepsilon_m] \mid \varepsilon_i \in \{0, 1\} \text{ et } \varepsilon_i \leq \varepsilon_{i+1}\}.$$

Pour $n \in \mathbb{N}^\star$, pair, et $x = [x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n(\mathbb{Q})$, notons

$$\delta := \gcd(x_1 - x_3, x_2, x_3 - x_5, x_4, \cdots, x_{n-1} - x_{n+1}, x_n).$$

Théorème 4.4.8. *Soit* n *un entier positif pair. Alors l'action* $B_{n+2} \curvearrowright \mathbb{P}^n(\mathbb{Q})$ *a une infinité d'orbites. Des invariants d'orbites sont donnés par* δ *et* $\pm x_1 \mod (\delta)$*.*

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Index

Étude des q-analogues de nombres réels : aspects combinatoires et géométriques, extensions, généralisation en dimension supérieure

Un analogue quantique est une déformation d'un objet mathématique réalisée par l'ajout d'un paramètre formel, traditionnellement noté q. La spécialisation $q \to 1$ doit redonner l'objet de départ, qui se voit ainsi plongé au sein d'une famille d'objets similaires. De nombreux objets ont des q-analogues naturels, et en particulier les entiers, dont les plus célèbres déformations remontent aux travaux d'Euler et Gauss sur les q-séries. En 2020, Morier-Genoud et Ovsienko ont introduit une déformation des nombres réels qui étend la notion de q-entier. Construits par des méthodes combinatoires, ces q-réels possèdent des propriétés inattendues en lien avec de nombreux domaines. L'objectif de cette thèse est double : poursuivre l'exploration de la théorie des q-nombres, et proposer des généralisations de cette théorie pour augmenter sa portée. Nous portons ici une attention particulière aux symétries modulaires régissant les q-réels, qui leurs confèrent une cohérence interne. Nous construisons algébriquement deux extensions successives de ces symétries, et nous les utilisons ensuite pour démontrer des résultats combinatoires sur les rationnels quantiques. Nous illustrons la richesse de la combinatoire des q-nombres en étudiant des analogues naturels des nombres de Markov, donnés par les traces de certaines matrices déformées. Nous changeons ensuite de point de vue en explorant l'effet de transformations géométriques sur les q-nombres, interprétés comme des disques dans le plan. Enfin, nous mettons à profit la théorie des représentations de tresses pour introduire une généralisation en dimension 2 des analogues quantiques de rationnels.

Mots-clés : Analogues quantiques, q-déformation, nombres de Markov, pavage de Farey, groupe modulaire, groupe de tresses, représentation de Burau, graphe en serpent.

Study of q-analogues of real numbers : combinatorial and geometric aspects, extensions, generalization in higher dimension

A quantum analogue is a deformation of a mathematical object by the addition of a formal parameter, tradionnally denoted by q. The specialization $q \to 1$ must give back the object, which is thus embedded in a family of similar objects. Many objects have natural q-analogues, in particular integers. The most famous deformation of integers goes back to Euler and Gauss who used it to study q-series. In 2020, Morier-Genoud and Ovsienko introduced a deformation of real numbers extending the notion of q-integers. Defined by combinatorial methods, these q-reals satisfy unexpected properties related to several areas of mathematics. The goal of this thesis is twofold : to continue the exploration of the theory of q-real numbers, and to define generalizations of this theory. We pay here particular attention to the modular symmetries underlying q-real numbers. We build algebraically two successive extensions of these symmetries, and we use them to prove combinatorial results on q-rational numbers. We illustrate the rich combinatorics of q-rationals by studying natural analogues of Markov numbers, given by the traces of some q-deformed matrices. We then shift our point of view and explore the effect of some geometric transformations on q-real numbers, that we interpret as disks in the plane. Finally, we take advantage of the theory of braid groups representations to introduce a dimension 2 generalization of quantum rational numbers.

Keywords : Quantum analogues, q-deformation, Markov numbers, Farey tesselation, modular group, braid group, Burau representation, snake graph.

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